The Dissertation Committee for Jonathan C. Johnson
certifies that this is the approved version of the following dissertation:

Two-Bridge Links, Pretzel Knots and Bi-Orderability

Committee:

Cameron Gordon, Supervisor

Jeffrey Danciger

John Luecke

Dale Rolfsen

Two-Bridge Links, Pretzel Knots and Bi-Orderability

by

Jonathan C. Johnson

DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

August 2021

Dedicated to my dad.

Acknowledgments

There is a multitude of people who have played a significant role during my time in graduate school. I want to thank the many faculty and staff who have helped me flourish at the University of Texas. In particular, I thank my advisor, Cameron Gordon, for all the suggestions, guidance, support, and encouragement he's given me while I've been working on this research project. I also thank John Luecke for all the time spent helping me learn Heegaard Floer homology. I want to thank Elisa Bass and Eva Hernandez for their continued support and friendship. The community of topology graduate students has been invaluable to me during this time. I thank Ahmad Issa, Allison Miller, and Lisa Piccirillo for their mentorship and guidance. I want to say a special thank you to my academic twin Hannah Turner for many math conversations, for reading my drafts, and for being a great friend. I also want to acknowledge all of the people who have made me the mathematician I am today. I thank my master's advisor, Barbara Shipman, and Mr. Everett, who in high school told me, "You're not an engineer. You're a mathematician." Finally, I want to acknowledge the many friends and family members who have loved and supported me on this amazing journey.

Two-Bridge Links, Pretzel Knots and Bi-Orderability

Publication No. ____________________

Jonathan C. Johnson, Ph.D.
The University of Texas at Austin, 2021

Supervisor: Cameron Gordon

The orderability of a 3-manifold group is closely connected to the topological properties of the manifold. Link groups are always left-orderable. However, there are link groups which are known to be bi-orderable, as well as link groups known not to be bi-orderable. In this dissertation, the bi-orderability of some families of link groups is shown. We show that two-bridge links with Alexander polynomials whose coefficients are coprime are extensions of $\mathbb{Z}$ by residually torsion-free nilpotent groups. It follows from a result of Linnell-Rhemtulla-Rolfsen[30] that if a two-bridge link has an Alexander polynomial with coprime coefficients and all real positive roots, then its link group is bi-orderable. In particular, if a two-bridge knot has an Alexander polynomial with all real positive roots, then its knot group is bi-orderable. This result shows that a large family of knots whose cyclic branched covers are known to be L-spaces have bi-orderable knot groups. Additionally, using a technique developed by Mayland, the pretzel knots $P(-3, 3, 2r+1)$ are shown to have bi-orderable knot groups. Issa-Turner[20] showed that all the cyclic branched covers of these knots are L-spaces. Finally, a family of genus one pretzel knots are shown to have bi-orderable knot groups and double branched covers which are not L-spaces.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Background

Let L be a smooth link(or knot) in S^3. Let M_L denote the *link exterior* $S^3 - \nu(L)$ where $\nu(L)$ is the interior of a tubular neighborhood of L. Also, let $\pi(L) := \pi_1(M_L)$ be the *link group* of L.

Definition 1.1. A group is *bi-orderable* if there is a total order of its elements, which is invariant under both left and right multiplication.

Several groups are known to be bi-orderable. Free groups are bi-orderable [33]. Abelian groups are bi-orderable if and only if they are torsion free [28]. Also, the free product of bi-orderable groups is bi-orderable [52].

Bi-orderability is closely related to left-orderability. A group is *left-orderable* when there is a total order of its elements invariant under left multiplication (but not necessarily right multiplication). Clearly, bi-orderability implies left-orderability. While the study of left-orderability for rational homology spheres is very rich with interesting questions, this is not true for 3-manifolds with positive first Betti number.

Theorem 1.2 (Howie-Short [19], Boyer-Rolfsen-Wiest [5])**.** *Let M be a compact connected orientable 3-maniold. If* $\mathrm{rk} H_1(M) = 0$*, then* $\pi_1(M)$ *is not* *bi-orderable. If* $\mathrm{rk}H_1(M) > 0$*, then* $\pi_1(M)$ *is left-orderable.*

In particular, link groups are always left-orderable. However, there are many examples of both bi-orderable and non-bi-orderable link groups.

Theorem 1.3 (Perron-Rolfsen [43])**.** *Let* J *be a fibered*[1] *knot. If all of the roots of the Alexander polynomial of* J *are real and positive, then* $\pi(J)$ *is bi-orderable.*

Remark 1.1.1. Though they did not state it, Perron and Rolfsen's proof also works for oriented fibered links.

In particular, Theorem 1.3 implies that the knot group of the figure-8 knot is bi-orderable.

Proposition 1.4. *If* L *is the* (m, n)*-torus link and* $m \neq n$*, then* $\pi(L)$ *is not bi-orderable.*

Proof. Suppose L is a (m, n)-torus link with $m \neq n$. Let $k = \gcd(m, n)$. Let $p = m/k$ and $q = n/k$. The group $\pi(L)$ has the following presentation [1, Theorem 4.3].

$$\langle a, b, \{f_i\}_{i=0}^{k} | f_0, \{a^p f_i b^{-q} f_i^{-1}\}_{i=0}^{k} \rangle$$

Since either $|p| > 1$ or $|q| > 1$, there exist some elements x and y such that x and y do not commute but $x^p = y^q$. By a result of Neumann [41], $\pi(L)$ is not bi-orderable. $\square$

[1]A link(or knot) L is *fibered* if there is an S^1 family of Seifert surfaces $\{F_t\}_{t\in S^1}$ such that $F_s \cap F_t = L$ when $s \neq t$.

In light of these two results, it is natural to seek a classification of bi-orderable link groups. Since Perron and Rolfsen's result, many more examples of links with and without bi-orderable groups have been found. Clay and Rolfsen found an obstruction to bi-orderability.

Theorem 1.5 (Clay-Rolfsen [11]). *If J is a fibered knot with bi-orderable knot group, then the Alexander polynomial of J has at least one real and positive root.*

This result has a particularly interesting implication, which connects bi-orderability to Osváth and Szabó's Heegaard Floer invariants.

Theorem 1.6 (Clay-Rolfsen [11]). *If J is a knot such that some non-trivial surgery on J is an L-space*[2]*, then $\pi(J)$ is not bi-orderable.*

Given an oriented link L, $\pi(L)$ is canonically an extension of $\mathbb{Z}$. Let $h : \pi(K) \to H_1(M_L)$ be the Hurewicz map, and let $\varphi : H_1(M_L) \to \mathbb{Z}$ be the map defined by identifying the homology classes of the oriented meridians of each component of L. Then, $\pi(K)$ is an extension of $\mathbb{Z}$ by $\ker(\varphi \circ h)$ as follows.

$$\begin{tikzcd} 1 \arrow[r] & \ker(\varphi \circ h) \arrow[r] & \pi(K) \arrow[r, "\varphi\circ h"] \arrow[d, "h"'] & \mathbb{Z} \arrow[r] & 1 \\ & & H_1(S^3 - K) \arrow[ur, "\varphi"'] & & \end{tikzcd} \tag{1.1}$$

Definition 1.7. The subgroup $\ker(\varphi \circ h)$ is called the *Alexander subgroup* of the oriented link L. When L is a knot, the Alexander subgroup is the commutator subgroup of $\pi(L)$.

[2] A rational homology sphere M is an *L-space* if $rk\widehat{HF}(M) = |H_1(M)|$.

Definition 1.8. A group G is *residually torsion-free nilpotent* if for every nontrivial element $x \in G$, there is a normal subgroup $N \lhd G$ such that $x \notin N$ and G/N is a torsion-free nilpotent group.

Let $f \in \mathbb{Q}[X]$ be a monic polynomial. Factor f into irreducible polynomials as follows.

$$f = f_1 \cdots f_n$$

We call f a *special polynomial* if for each i

- f_i has all real positive roots or
- f_i has odd prime power degree, negative constant term, and all real roots
  .

The following theorem, which is a result of work by Linnell, Rhemtulla and Rolfsen, has proven to be an extremely useful for finding examples of links with bi-orderable link groups.

Theorem 1.9 (Linnell-Rhemtulla-Rolfsen [30])**.** *Let L be an oriented link in S^3. If the Alexander subgroup of L is residually torsion-free nilpotent and the Alexander polynomial of L is a scalar multiple of a special polynomial, then $\pi(K)$ is bi-orderable.*

Suppose that some link group has a presentation with two generators and one relation. Chiswell, Glass, and Wilson found that if the relation satisfies certain conditions, the link group is bi-orderable. They also found an obstruction for the bi-orderability of link groups.

Let w be a word in the free group generated by x and t such that the exponent sum of t in w is zero. Denote the conjugate of x by y by $x^y := y^{-1}xy$, The word w can written

$$w = (x^{m_1})^{t^{d_1}} \dots (x^{m_n})^{t^{d_n}}$$

for some positve integer n and some integers $d_1, \dots, d_n, m_1, \dots, m_n$. For each integer j, let

$$\tau_j(w) = \{i : d_i = j\},$$

and let

$$S_w = \{j : \sum_{i \in \tau_j(w)} m_i \neq 0\}$$

Suppose that S_w is not empty. Denote by f_w and e_w the maximum and minimum integers in S_w. A word w is *tidy* if S_w is nonempty and $\tau_j(w)$ is empty for all $j > f_w$ and $j < e_w$. If w is tidy and $|\tau_{f_w}(w)| = 1$, then w is called *principal*.

Theorem 1.10 (Chiswell-Glass-Wilson [9]). *Suppose L is a link whose link group has a presentation with two generators and one tidy relation w.*

(a) If w is principal and the coefficients of the Alexander polynomial of L are relatively prime(collectively, not pairwise), then $\pi(L)$ is bi-orderable.

(b) If $\pi(L)$ is bi-orderable then the Alexander polynomial of L has at least one real positive root.

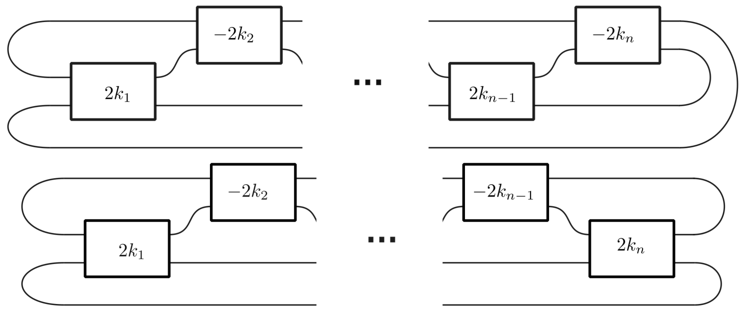


Figure 1.1: Rational tangle form of a two-bridge knot (top) and link (bottom). The integers indicate number of right-handed half-twist when positive and left-handed half-twist when negative.

Definition 1.11. A link L is a *two-bridge* link if it is the closure of a rational tangle.

For each oriented two-bridge link L, there are integers $k_1, \ldots, k_n$ such that L has a diagram as in Figure 1.1. Let $L(2k_1, \ldots, 2k_n)$ denote the two-bridge link defined by this diagram.

Clay, Desmarais, and Naylor showed that two-bridge link groups have presentations with two generators and one tidy relation. This produced bi-orderable and non-bi-orderable families of two-bridge knot groups.

Theorem 1.12 (Clay-Desmarais-Naylor [10])**.** *Suppose J is a two-bridge knot.*

1. *If $\pi(J)$ is bi-orderable, then the Alexander polynomial of J has at least one real and positive root.*

2. *If J is an even twist knot, $L(2, 2k)$ with $k > 0$, then $\pi(J)$ is bi-orderable.*

These results have been extended by Ito and Yamada.

Theorem 1.13 (Yamada [53]). *Suppose L is the two bridge link $L(2, \ldots, 2, 2k)$ with $k > 0$, then $\pi(L)$ is bi-orderable.*

Theorem 1.14 (Ito [21]). *Suppose J is any knot such that the degree of Alexander polynomial of J is twice its Seifert genus[3]. If $\pi(J)$ is bi-orderable, then the Alexander polynomial of J has at least one real and positive root.*

1.2 Summary of Results

Baumslag's work on parafree groups [2, 3] provides a sufficient condition for a group to be residually torsion-free nilpotent. In a talk, Mayland [34] proposed a strategy to show that the commutator subgroup of a two-bridge knot satisfies Baumslag's conditions. Is chapter 5, we implement Mayland's strategy to show that the Alexander subgroups of all two-bridge links whose Alexander polynomials have relatively prime coefficients satisfy Baumslag's conditions, which implies the following result.

Theorem A (Johnson [24]). *Let L be an oriented two-bridge link. If the Alexander polynomial of L has relatively prime coefficients and all real and positive roots, then $\pi(L)$ is bi-orderable. In particular, if J is a two-bridge knot and all the roots of the Alexander polynomial of J are real and positive, then the knot group of J is bi-orderable.*

---

[3]The *Seifert genus* of a knot J is the minimal genus of a surface embedded in S^3 bounded by J.

Lyubich and Murasugi found sufficient conditions for all the roots of the Alexander polynomial of a two-bridge link L to be real and positive.

Theorem 1.15 (Lyubich-Murasugi [32])**.** *Suppose L is the oriented two-bridge link $L(2k_1, \ldots, 2k_n)$ with $k_i > 0$ for each $i = 1, \ldots, n$, then all the roots of $\Delta_L(t)$ are real and positive.*

Combining this theorem with Theorem A implies the following.

Corollary 1.16. *Suppose L is the oriented two-bridge link $L(2k_1, \ldots, 2k_n)$ with $k_i > 0$ for each $i = 1, \ldots, n$.*

If the coefficients of the Alexander polynomial of L are relatively prime, then the link group of L is bi-orderable. In particular, when L is a knot, the knot group of L is bi-orderable.

Theorem 1.15 does not characterize all two-bridge links with Alexander polynomial that have all real and positive roots. The following example was brought to our attention by Ahmad Issa.

Example 1.17. *Let $K = L(2, 2, -8, -2)$.*

$$\Delta_K(t) = 4t^4 - 20t^3 + 33t^2 - 20t + 4 = (t-2)^2(2t-1)^2$$

which has two real roots of multiplicity 2. Thus, the knot group of K is bi-orderable.

Given an oriented link L in S^3 and an integer $d \geqslant 2$, let $\Sigma_d(L)$ denote the *d-fold branched cyclic cover of J*, the manifold which is the unique d-fold cyclic covering of S^3 with banching set L. It is common to call $\Sigma_2(L)$ the *double branched cover* of L.

Definition 1.18. An oriented link is called a *branched L-space link* if every $\Sigma_d(L)$ is an L-space for all $d \geqslant 2$.

Theorem 1.19 (Teragaito [50])**.** *The two bridge links $L(2k_1, \ldots, 2k_n)$ with $k_i > 0$ for each $i = 1, \ldots, n$ are branched L-space links.*

Suppose the knot J is equal to $L(2k_1, \ldots, 2k_n)$ with $k_i > 0$ for all i. Theorem A shows J has bi-orderable knot groups, and Theorem 1.19 shows J is a branched L-space knot. These two results motivated the following question.

Question 1.20. *Is it true that for any knot J, $\pi(J)$ is bi-orderable if and only if J is a branched L-space knot?*

After two-bridge knots, the next class of knots we consider are the genus one *pretzel knots* $P(2p+1, 2q+1, 2r+1)$ pictured in Figure 1.2. In chapter 6, we find examples of genus one pretzel knots with and without bi-orderable knot groups.

Theorem B (Johnson [23])**.** *Let J be the $P(-3, 2q+1, 2r+1)$ pretzel knot with $1 \leqslant q \leqslant r$.*

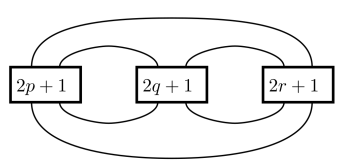


Figure 1.2: A genus one pretzel knot

(a) If J is $P(-3,3,2r+1)$, then $\pi(J)$ is bi-orderable.

(b) If J is $P(-3,5,2r+1)$ with $r>3$ or $P(-3,2q+1,2r+1)$ with $q\geqslant 2$, then $\pi(J)$ is not bi-orderable.

Remark 1.2.1. Theorem B accounts for all the $P(-3,2q+1,2r+1)$ pretzel knots except $P(-3,5,5)$ and $P(-3,5,7)$.

The $P(-3,3,2r+1)$ pretzel knots are the first knots which aren't alternating or fibered known to have bi-orderable knot groups.

Suppose J is a pretzel knot $P(2p+1,2q+1,2r+1)$ with $1\leqslant q\leqslant r$, and $p\leqslant -2$. By the Montesinos trick [37], the double branched cover of J is the Seifert fibered space

$$\Sigma_2(J) = M(0;-1,\frac{-2p-2}{-2p-1},\frac{1}{2q+1},\frac{1}{2r+1}). \tag{1.2}$$

By work of Boyer-Gordon-Watson [4], Eisenbud-Hirsch-Neumann [13], Jankins-Neumann [22], Lisca-Stipsicz [31], Naimi [40], and Ozsváth-Szabó [42], it is known precisely when a Seifert fibered space is an L-space. In particular, we have the following result.

Theorem 1.21 (Eisenbud-Hirsch-Neumann [13], Jankins-Neumann [22], Lisca-Stipsicz [31], Naimi [40])**.** *Let M be the Seifert fibered space*

$$M(0;-1,\frac{\beta_1}{\alpha_1},\frac{\beta_2}{\alpha_2},\frac{\beta_3}{\alpha_3})$$

where α_j and β_j are integers for which $0<\beta_j<\alpha_j$. M is not an L-space if and only if there are relatively prime integers $0<a<m$ such that for some

permutation (r_1, r_2, r_3) *of the triple* $(\frac{a}{m}, \frac{m-a}{m}, \frac{1}{m})$ *we have that* $\frac{\beta_j}{\alpha_j} < r_j$ *for each* $j = 1, 2, 3$.

Corollary 1.22. *Suppose* J *is the* $P(2p+1, 2q+1, 2r+1)$ *pretzel knot with* $p \leqslant -2$ *and* $1 \leqslant q \leqslant r$. $\Sigma_2(J)$ *is an L-space if and only if* $-p > q$.

Proof. By (1.2), the double branched cover of J is $M(0; -1, \frac{-2p-2}{-2p-1}, \frac{1}{2q+1}, \frac{1}{2r+1})$. Suppose $-p \leqslant q$. Let $m = 2q$ and $a = 2q - 1$.

Since $-p \leqslant q$, $-2p - 1 < 2q$ so

$$\frac{-2p-2}{-2p-1} = 1 - \frac{1}{-2p-1} < 1 - \frac{1}{2q} = \frac{2q-1}{2q}.$$

Thus, the triple $(\frac{-2p-2}{-2p-1}, \frac{1}{2q+1}, \frac{1}{2r+1})$ is less than $(\frac{2q-1}{2q}, \frac{1}{2q}, \frac{1}{2q})$. Therefore, by Corollary 1.22, $\Sigma_2(J)$ is an not an L-space.

Suppose $-p > q$. Let m and a be positive integers such that $0 < a < m$. The sum of any two of $\frac{a}{m}$, $\frac{m-a}{m}$, or $\frac{1}{m}$ is at most 1. Since $-p > q$ and both p and q are integers, $-p - \frac{1}{2} \geqslant q + \frac{1}{2}$ so $-2p - 1 \geqslant 2q + 1$. Thus,

$$\frac{-2p-2}{-2p-1} + \frac{1}{2q+1} = 1 - \frac{1}{-2p-1} + \frac{1}{2q+1} \geqslant 1 - \frac{1}{2q+1} + \frac{1}{2q+1} = 1.$$

Therefore, no two of $\frac{a}{m}$, $\frac{m-a}{m}$ or $\frac{1}{m}$ are bigger than the pair $(\frac{2q-1}{2q}, \frac{1}{2q})$. It follows, by Corollary 1.22, that $\Sigma_2(J)$ is an L-space. $\square$

All of the knots in Theorem B(b) don't have bi-orderable knot groups, and by Corollary 1.22, they are not branched L-space knots. By B(b), the $P(-3, 3, 2r+1)$ pretzel knots have bi-orderable knot groups. Issa and Turner [20] confirmed that these knots are branched L-space knots.

Naturally, we move on to the pretzel knots $P(-5, 2q+1, 2r+1)$ where we find that $P(-5, 7, 9)$ has a bi-orderable knot group and a double branched cover which isn't an L-space. In fact, we show that there are infinitely many of such examples in chapter 6.

Theorem C (Johnson [23])**.** *For each integer $q \geqslant 3$, let J_q be the $P(1-2q, 2q+1, 4q-3)$ pretzel knot. When $q-1$ is a prime power, $\pi(J_q)$ is bi-orderable, and $\Sigma_2(J_q)$ is not an L-space. In particular, when $q \geqslant 3$, J_q is not a branched L-space knot.*

Chapter 2

Preliminaries

2.1 Preliminaries on Presentation Matrices

Let R be a PID. Suppose X is an R-module with presentation

$$\langle x_1, \ldots, x_n | s_1, \ldots, s_m \rangle.$$

For each i,

$$s_i = \sum_{j=1}^{n} r_{i,j} x_j$$

where each $r_{i,j}$ is in R. The matrix of $r_{i,j}$ coefficients

$$\begin{pmatrix} r_{1,1} & \cdots & r_{1,n} \\ \vdots & & \vdots \\ r_{m,1} & \cdots & r_{m,n} \end{pmatrix}$$

is called a *presentation matrix* of X.

Suppose A is a presentation matrix of X. Performing row and column operations on A will always produce another presentation matrix of X. In particular, using row and column operations, A can be diagonalized into the following form

$$\left(\begin{array}{ccc|c} d_1 & & & \\ & \ddots & & 0 \\ & & d_k & \\ \hline & 0 & & 0 \end{array}\right)$$

where each d_i is nonzero and d_i divides d_{i+1} for each $i = 1, \ldots, k-1$. Therefore,

$$X \cong R^{n-k} \oplus \frac{R}{d_1 R} \oplus \cdots \oplus \frac{R}{d_k R}. \tag{2.1}$$

The d_i which are not units are the invariant factors of X.

The following lemma plays a key role in showing that elements in a parafree group are homologically primitive.

Lemma 2.1. *Suppose X is an R-module with an $m \times n$ presentation matrix A of full rank. If the greatest common divisor of every $m \times m$ minor of A is a unit, then X is a free R-module. Otherwise, the greatest common divisor of every $m \times m$ minor of A is equal to the product of the invariant factors of X up to multiplication by a unit.*

Proof. Let B be A after diagonalization. Since A has full rank, B has no extra rows of zeros so B has the following form.

$$B = \left(\begin{array}{ccc|c} d_1 & & & \\ & \ddots & & 0 \\ & & d_m & \end{array} \right)$$

For any $m \times n$ matrix with entries in R, the greatest common divisor of its $m \times m$ minors is invariant under row and column operations up to multiplication by a unit. Therefore, up to a unit, the greatest common divisor of the $m \times m$ minors of A is $\prod_{i=1}^{m} d_i$. When $\prod_{i=1}^{m} d_i$ is a unit, each d_i is a unit so by (2.1), X is a free R-module. If $\prod_{i=1}^{m} d_i$ is not a unit, it is the product of the invariant factors of X. $\square$

2.2 Bi-orders, Extensions, and Link Groups

Group extensions provide an effective way to construct or obstruct the bi-orderability of a group.

2.2.1 Bi-Orderability and Extensions of $\mathbb{Z}$

Suppose that a group G is an extension of $\mathbb{Z}$ by some subgroup Y as follows.

$$1 \longrightarrow Y \longrightarrow G \xrightarrow{f} \mathbb{Z} \longrightarrow 1$$

Suppose $m \in f^{-1}(1)$.

Proposition 2.2. *G is bi-orderable if and only if there is a bi-ordering of Y invariant under conjugation by m.*

Proof. If G is bi-orderable, then the bi-order on G restricts to a bi-order of Y invariant under conjugation by elements of G.

Suppose there is a bi-order $<_Y$ such that for all $x, y \in Y$, $x <_Y y$ if and only if $m^{-1}xm <_Y m^{-1}ym$. For every element g of G, let y_g be unique element in Y such that $g = m^{f(g)}y_g$. Define a total order on G as follows. For each pair of elements a and b in G, let $a < b$ if and only if $f(a) < f(b)$ or $f(a) = f(b)$ and $y_a <_Y y_b$.

Suppose $a < b$, and let g be an arbitrary element of G. If $f(a) < f(b)$, then

$$f(ag) = f(ga) = f(a) + f(g) < f(b) + f(g) = f(bg) = f(gb).$$

It follows that $ga < gb$ and $ag < bg$.

Suppose $f(a) = f(b)$. Thus, since $a < b$, $y_a <_Y y_b$, and since $f(a) = f(b)$, $f(ga) = f(gb)$ and $f(ag) = f(bg)$.

Let $y'_g = m^{f(g)} g m^{-f(g)}$.

$$\begin{aligned} ga =& m^{f(g)} y_g m^{f(a)} y_a \\ =& m^{f(g)+f(a)} m^{-f(a)} y_g m^{f(a)} y_a \\ =& m^{f(g)+f(b)} m^{-f(b)} y_g m^{f(b)} y_a \end{aligned}$$

Since $m^{-f(b)} y_g m^{f(b)} y_a <_Y m^{-f(b)} y_g m^{f(b)} y_b$,

$$ga = m^{f(g)+f(b)} m^{-f(b)} y_g m^{f(b)} y_a < m^{f(g)+f(b)} m^{-f(b)} y_g m^{f(b)} y_b = gb$$

so $<$ is invariant under left multiplication.

$$\begin{aligned} ag =& m^{f(a)} y_a m^{f(g)} y_g \\ =& m^{f(g)+f(a)} m^{-f(g)} y_a m^{f(g)} y_g \\ =& m^{f(g)+f(b)} m^{-f(g)} y_a m^{f(g)} y_g \end{aligned}$$

Since $<_Y$ is invariant under conjugation by m,

$$m^{-f(g)} y_a m^{f(g)} y_g <_Y m^{-f(g)} y_b m^{f(g)} y_g.$$

Thus,

$$ag = m^{f(g)+f(b)} m^{-f(g)} y_a m^{f(g)} y_g < m^{f(g)+f(b)} m^{-f(g)} y_b m^{f(g)} y_g = bg$$

so $<$ is invariant under right multiplication. $\square$

Suppose L is an oriented n-component link. As we've seen in chapter 1, $\pi(L)$ is canonically an extension of $\mathbb{Z}$ by the Alexander subgroup. However, there are multiple ways to extend $\mathbb{Z}$ to $\pi(L)$ as follows.

$$\begin{tikzcd} 1 \arrow[r] & \ker(f \circ h) \arrow[r] & \pi(L) \arrow[r, "f \circ h"] \arrow[d, "h"'] & \mathbb{Z} \arrow[r] & 1 \\ & & H_1(S^3 - L) \arrow[ur, "f"'] & & \end{tikzcd}$$

Here h is the Hurewicz map, and f can be any surjective map from $H_1(M_L)$ to $\mathbb{Z}$. Applying Proposition 2.2 to this situation yields the following useful fact.

Proposition 2.3. *Suppose L is an oriented n-component link, and let f be any surjective map from $H_1(M_L)$ to $\mathbb{Z}$. The link group $\pi(L)$ is bi-orderable if and only if there is an order on* $\ker(f \circ h)$ *invariant under conjugation by an element in $(f \circ h)^{-1}(1)$.*

2.2.2 Bi-Orderability and Connect Sums

Using Proposition 2.3 and the following result of Rolfsen [46], we see that bi-orderability behaves well under connect sum.

Theorem 2.4 (Rolfsen [46])**.** *Let G_i be a group with bi-order $<_i$ for $i = 1, 2$. There exist a bi-order $\prec$ on $G_1 * G_2$ such that if $\phi_i \in \mathrm{Aut}(G_i)$ preserves the bi-order $<_i$ for each $i = 1, 2$, then $\phi_1 * \phi_2$ preserves $\prec$.*

Proposition 2.5. *Let L_1 and L_2 be smooth oriented links in S^3.*

$\pi(L_1 \# L_2)$ is bi-orderable if and only if $\pi(L_i)$ is bi-orderable for $i = 1, 2$.

Proof. Let L_1 and L_2 be smooth oriented links in S^3. Let $L = L_1 \# L_2$ which is obtained by summing a component of L_1 with a component of L_2.

Consider the maps $f_i : H_1(M_{L_i}) \to \mathbb{Z}$ defined by sending the meridian of the summed component to 1 and sending the meridians of all the other components to 0. Define ϕ_i to be $f_i \circ h$, and let $G_i = \ker(\phi_i)$ for $i = 1, 2$. Since the short exact sequence of an extension of $\mathbb{Z}$ always splits,

$$\pi(L_i) \cong G_i \rtimes_{\phi_i} \langle t_i \rangle$$

for $i = 1, 2$.

By Seifert-van Kampen's theorem, $\pi_1(L)$ is the free product of $\pi(L_1)$ and $\pi(L_2)$ after amalgamating the generators of the $\mathbb{Z}$ factors. By the universal property of free products with amalgamation,

$$\pi(L) \cong G_1 * G_2 \rtimes_{\phi_1 * \phi_2} \mathbb{Z}$$

For each $i = 1, 2$, $\pi(L_i)$ embeds into $\pi(L)$ so if $\pi(L)$ is bi-orderable, $\pi(L_i)$ is bi-orderable.

Suppose $\pi(L_i)$ is bi-orderable of each $i = 1, 2$. By Proposition 2.3, there is a bi-order $<_i$ on G_i invariant under ϕ_i, and by Theorem 2.4, there is a bi-order $\prec$ on $G_1 * G_2$ invariant under $\phi_1 * \phi_2$. Therefore, by Proposition 2.3, $\pi(L)$ is bi-orderable. $\square$

2.3 Residual Torsion-Free Nilpotence

2.3.1 Baumslag's Residual Torsion-Free Nilpotence Conditions

Baumslag's work on parafree groups [2, 3] provides a sufficient condition for a group to be residually torsion-free nilpotent.

Definition 2.6. Let G be a group. Define $\gamma_1 G := G$, and for each positive integer n, define $\gamma_{n+1} G := [G, \gamma_n G]$. A group G is *parafree of rank r* if

1. for some free group F of rank r, $G/\gamma_n G \cong F/\gamma_n F$ for each n, and

2. G is residually nilpotent.

Proposition 2.7 (Baumslag [3, Proposition 2.1(i)]). *Suppose G is a group which is the union of an ascending chain of subgroups as follows.*

$$G_0 < G_1 < G_2 < \cdots < G_n < \cdots < G = \bigcup_{n=1}^{\infty} G_n$$

Suppose each G_n is parafree of the same rank. If for each non-negative integer n, $|G_{n+1} : G_n[G_{n+1}, G_{n+1}]|$ is finite then G is residually torsion-free nilpotent.

Definition 2.8. Let H be a parafree group of rank r. An element $h \in H$ is *homologically primitive* if the class of h in $H/[H, H] \cong \mathbb{Z}^r$ can be extended to a basis.

Proposition 2.9 (Baumslag [2, Proposition 3]). *Let H be a parafree group of rank r, and let $\langle t \rangle$ be an infinite cyclic group generated by t. Let h be an element in H, and n be a positive prime integer. If h generates its own centralizer and h is homologically primitive in H, then the group*

$$H \underset{h=x^n}{*} \langle x \rangle$$

is parafree of rank r.

A theorem of Baumslag [3, Theorem 4.2] states that any two-generator subgroup of a parafree group is free. If follows that an element homologically primitive in a parafree group must generate its own centralizer.

Suppose n from Proposition 2.9 is composite, and let $n = p_1 \cdots p_k$, be the prime decomposition of n, and define

$$G_j = \langle H * \langle x_1 \rangle * \cdots * \langle x_j \rangle \mid h = x_1^{p_1}, x_1 = x_2^{p_2}, \ldots, x_{j-1} = x_j^{p_j} \rangle$$

for $j = 1, \ldots, k$ so

$$G_k \cong H \underset{h=x^n}{*} \langle x \rangle.$$

For each $j = 1, \ldots, k-1$, x_j is homologically primitive in G_j. Therefore, Proposition 2.9 is strengthened to the following statement.

Proposition 2.10. *Let H be a parafree group of rank r, and let $\langle x \rangle$ be an infinite cyclic group generated by x. Let h be an element in H, and n be any positive integer. If h is homologically primitive in H, then*

$$H \underset{h=x^n}{*} \langle x \rangle$$

is parafree of rank r.

2.3.2 Trivial Alexander Polynomial Obstruction

The following proposition provides an obstruction to the Alexander subgroup of a link being residually torsion-free nilpotent.

Proposition 2.11. *If L is an oriented link in S^3 with trivial Alexander polynomial, then the Alexander subgroup of L cannot be residually torsion-free nilpotent.*

Proof. Let L be an oriented link with Alexander polynomial Δ_L. Let Y be the Alexander subgroup of L. Let M^∞ be the infinite cyclic cover of L, the covering space of M_L corresponding to to subgroup Y; see [45, Chapter 7] for details.

$$H_1(M^\infty, \mathbb{Z}) \cong \bigoplus_{i=1}^{n} \mathbb{Z}[t, t^{-1}]/\langle a_i(t) \rangle$$

where $a_1(t), \ldots, a_n(t)$ are polynomials such that

$$\prod_{i=1}^{n} a_i(t) = \Delta_L(t).$$

Since the Alexander polynomial of L is trivial, $Y/[Y,Y] \cong H_1(M^\infty) = 1$ so $Y = [Y,Y]$. It follows that every term, Y_i, of the lower central series of Y is isomorphic to Y. Suppose $N \lhd Y$ is a proper normal subgroup of Y. For each term, $(Y/N)_i$, of the lower central series of Y/N,

$$(Y/N)_i \cong Y_i/N \cong Y/N \neq 1$$

so Y/N cannot be nilpotent. Thus, Y is not residually torsion-free nilpotent. $\square$

2.4 The Reidemeister-Schreier Rewriting Procedure

Suppose G is a group with presentation $\langle x_1 \ldots, x_n | r_1, \ldots, r_m \rangle$. Let H be a subgroup of G. Reidemeister [44] and Schreier [47], developed a proceedure to obtain a group presentation of H. Section 2.3 of the text by Karrass, Magnus, and Solitar [25] describes the procedure in detail, but we will outline how it works here.

Let $\mathcal{C}$ be a set of right coset representatives for G/H satisfying the following condition: given any element $c = y_1 \cdots y_k \in \mathcal{C}$ where each y_i is a generator x_j, every initial segment $c_i = y_1 \cdots y_i$ is in $\mathcal{C}$ for all $i = 0, \ldots, k$ (here c_0 is the identity element). Given an element g in G, let $\overline{g}$ be the coset representative of g in $\mathcal{C}$. For each $x \in \{x_1, \ldots, x_n\}$ and $c \in \mathcal{C}$, define

$$\gamma(c, x) := cx(\overline{cx})^{-1}.$$

Define a generating set $\mathcal{S}$ of H as follows.

$$\mathcal{S} := \{\gamma(c, x) \neq 1 | x \in \{x_1, \ldots, x_n\}, c \in \mathcal{C}\}$$

Consider a word $u = y_1^{s_1} y_2^{s_2} \cdots y_k^{s_k}$ with $y_i \in \{x_1, \ldots, x_n\}$ and $s_i = \pm 1$ for all i. When u is in H, we can rewrite u in the generating set $\mathcal{S}$ as follows. Define

$$\tau(u) := \gamma(\overline{t_1}, x_1)^{s_1} \gamma(\overline{t_2}, x_2)^{s_2} \cdots \gamma(\overline{t_n}, x_n)^{s_n}$$

where

$$t_i := \begin{cases} x_1^{s_1} \cdots x_{i-1}^{s_{i-1}} \text{ (possibly trivial)}, & s_i = 1 \\ x_1^{s_1} \cdots x_i^{s_i}, & s_i = -1 \end{cases}.$$

Define a set of relations $\mathcal{R}$ for H from the relations $r_1 \ldots, r_m$ as follows.

$$\mathcal{R} := \{c r_i c^{-1} | c \in \mathcal{C}, i \in \{1, \ldots, m\}\}$$

Proposition 2.12 (Karrass-Magnus-Solitar [25, Theorem 2.9])**.**

$$H \cong \langle \mathcal{S} \mid \mathcal{R} \rangle$$

Chapter 3

Alexander Subgroups of Two-Bridge Links

3.1 Two-bridge links and Conway Notation

Every oriented two-bridge link is the closure of a rational tangle. Thus, by Conway's correspondence, we can associate a two-bridge link to a rational fraction p/q with $p > 0$; see [8, Chapter 12] for details. Let $L(p/q)$ denote the two-bridge link represented by p/q. Choose an orientation of $L(p/q)$ so that the two overstrands of Schubert's projection of $L(p/q)$ are oriented away from each other as in Figure 3.1. This correspondence satisfies the following properties:

1. $L(p/q)$ and $L(p'/q')$ are equivalent as unoriented links if and only if
   (a) $p = p'$ and
   (b) $q \cong q' \ (\mathsf{mod}\, p)$ or $qq' \cong 1 \ (\mathsf{mod}\, p)$.
2. $L(p/q)$ and $L(p'/q')$ are equivalent as oriented links if and only if
   (a) $p = p'$ and
   (b) $q \cong q' \ (\mathsf{mod}\, 2p)$ or $qq' \cong 1 \ (\mathsf{mod}\, 2p)$.
3. $L(p/q)$ is a knot if and only if p is odd.
4. $L(p/q)$ and $L(-p/q)$ are mirrors.

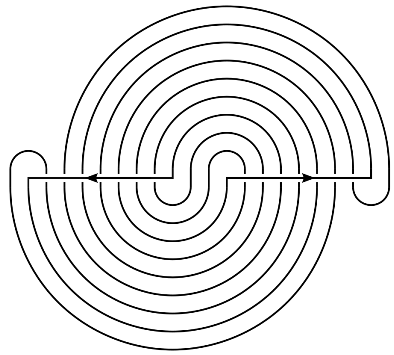


Figure 3.1: Schubert's projection of $L(8/3)$.

5. If $L(p/q)$ is a link, $L(p/(q \pm p))$ is the oriented link obtained by reversing the orientation of one of the components of $L(p/q)$.

When $p > |q| > 0$ and q is odd, there are non-zero integers $k_1, \ldots, k_n$ such that $p/(p-q) = [2k_1, \ldots, 2k_n]$, and $L(p/q) = L(2k_1, \ldots, 2k_n)$. Here $[2k_1, \ldots, 2k_n]$ denotes the continued fraction expansion

$$[2k_1, \ldots, 2k_n] = 2k_1 + \cfrac{1}{2k_2 + \frac{1}{2k_3 + \frac{1}{\cdots + \frac{1}{2k_n}}}}.$$

For details on fraction expansions and rational tangles, see [39, Chapter 9]. When n is even, $L(p/q)$ is a knot with genus $n/2$. When n is odd, $L(p/q)$ is a two-component link with genus $(n-1)/2$.

3.2 A Presentation of the Alexander Subgroup

In this section, we give a group presentation of the Alexander subgroup of an arbitrary two-bridge link group using the Reidemeister-Schreier rewriting process. From this presentation of the Alexander subgroup, we can describe the subgroup as the union of an ascending chain of subgroups which satisfies Baumslag's conditions when the Alexander polynomial of the link has relatively prime coefficients.

Consider the 2-bridge link $L := L(p/q)$ where $1 \leqslant |q| < p$ with q odd. For each integer i, define

$$\epsilon_i := (-1)^{\lfloor \frac{iq}{p} \rfloor}. \tag{3.1}$$

Proposition 3.1 (Schubert [48]). *Given the 2-bridge link $L(p/q)$,*

$$\pi(L(p/q)) \cong \langle a, b | w \rangle$$

where $w = a^{\epsilon_0} b^{\epsilon_1} \dots a^{\epsilon_{2p-2}} b^{\epsilon_{2p-1}}$.

Let Y be the Alexander subgroup of L. Y is the kernel of the map $\pi(L(p/q)) \to \langle t \rangle$ which sends both a and b to t. A group presentation for Y can be obtained using the Reidemeister-Schreier rewriting procedure.

Consider $\mathcal{A} := \{a^k\}_{k \in \mathbb{Z}}$ as a set of coset representatives for $\pi(L)/Y$. Note that $\gamma(a^k, a) = 1$, and $\gamma(a^k, b) = a^k b a^{-k-1}$. For each integer k, define

$$S_k := \gamma(a^k, b).$$

We have to following set of generators of Y from the Reidemeister-Schreier.

$$\mathcal{S} := \{S_k\}_{k \in \mathbb{Z}}$$

Since, for all k, $\gamma(a^k, a) = 1$, for each word u, $\tau(u)$ is a product $S_{k_1} S_{k_2} \cdots S_{k_l}$. For each integer k, define

$$R_k := \tau(a^k w a^{-k}).$$

Define

$$\sigma_i := \begin{cases} \sum_{j=0}^{i-1} \epsilon_j & \text{when } i > 0 \\ \sum_{j=i}^{-1} \epsilon_j & \text{when } i < 0 \\ 0 & \text{when } i = 0 \end{cases} \tag{3.2}$$

for each integer i.

Proposition 3.2. *Suppose $R_0 = \tau(w) = S_{i_1}^{\eta_1} S_{i_2}^{\eta_2} \ldots S_{i_n}^{\eta_n}$ where each i_j is an integer and each η_j is ± 1. Then,*

(a) $n = p$,

(b) $\eta_j = \epsilon_{2j-1}$, for each $j = 1, \ldots, p$,

(c) $i_j = \sigma_{2j-1}$ if $\eta_j = 1$ and $i_j = \sigma_{2j}$ if $\eta_j = -1$ for each $j = 1, \ldots, p$, and

(d) for every integer k, $R_k = S_{i_1+k}^{\eta_1} S_{i_2+k}^{\eta_2} \ldots S_{i_p+k}^{\eta_p}$.

Proof. Since $\gamma(a^k, a)$ is trivial, the S_i-generators in R_0 come from the b-generators in w. For (a), notice that the length of the word R_0 is the number of times b and b^{-1} appear in w which is equal to p. By definition η_j is equal to the exponent of the corresponding b or b^{-1} in w which is ϵ_{2j-1} showing (b). Since $a = b$ modulo Y, then for any word u in a and b, $\overline{u} = a^s$ where s is the sum of the exponents of the a's and b's in u. Thus, both (c) and (d) follow by a straightforward computation. □

By Proposition 2.12 we have the following.

Proposition 3.3.

$$Y \cong \langle \{S_k\}_{k \in \mathbb{Z}} \mid \{R_k\}_{k \in \mathbb{Z}} \rangle$$

3.3 Group Presentation Properties

This group presentation of Y has a few notable properties which will be of use.

Given a word W in $\mathcal{S}$, let $[W]$ denote the class of W in the free abelian group generated by $\mathcal{S}$. For each integer k, define $S'_k := [S_k]$. Denote the maximal and minimal subscripts of S appearing in the word R_0 by M and m respectively so that

$$[R_0] = a_M S'_M + a_{M-1}S'_{M-1} + \cdots + a_{m+1}S'_{m+1} + a_m S'_m.$$

for some integers $a_m, \ldots, a_M$.

Proposition 3.4. *Suppose L is a two-bridge link, and suppose Y is the Alexander subgroup of L with presentation as defined in section 3.2.*

(a) For each integer n,

$$[R_n] = a_M S'_{M+n} + a_{M-1}S'_{M-1+n} + \cdots + a_{m+1}S'_{m+1+n} + a_n S'_{m+n}.$$

(b) Let g be the genus of L. When L is a knot, $M - m = 2g$, and when L is a link, $M - m = 2g + 1$.

(c) For all $j = m, \ldots, M$

$$a_j = \begin{cases} \underline{a}_{g+m-j} & \text{if } m \leqslant j \leqslant m+g \\ \underline{a}_{g+j-M} & \text{if } M-g \leqslant j \leqslant M \end{cases}$$

where

$$\Delta_L(t) = \underline{a}_g t^{2g} + \cdots + \underline{a}_0 t^g + \cdots + \underline{a}_g$$

when L is a knot, and

$$\Delta_L(t) = \underline{a}_g t^{2g+1} + \cdots + \underline{a}_0 t^{g+1} + \underline{a}_0 t^g + \cdots + \underline{a}_g$$

when L is a link. In particular, for all $j = 0, \ldots, M-m$,

$$a_{M-j} = a_{m+j}.$$

Proof. Part (a) follows from Proposition 3.2(d).

For each $i = 1, \ldots, 2p$, denote by w_i the word obtained from the first i generators of the relation w. Also, define

$$\theta(s) := \begin{cases} 1 & \text{if } s = 1 \\ 0 & \text{if } s = -1 \end{cases}.$$

We compute the Alexander polynomial by performing Fox calculus on w with respect to b (see [14, Section 3]),

$$\begin{aligned}
\frac{\partial w}{\partial b} &= a^{\epsilon_0}\Big(\frac{\partial}{\partial b}(b^{\epsilon_1}) + b^{\epsilon_1}a^{\epsilon_2}\Big(\frac{\partial}{\partial b}(b^{\epsilon_3}) + \cdots + b^{\epsilon_{2p-3}}a^{\epsilon_{2p-2}}\Big(\frac{\partial}{\partial b}(b^{\epsilon_{2p-1}})\Big)\\
&= \sum_{i=1}^{p} w_{2i-1}\frac{\partial}{\partial b}(b^{\epsilon_{2i-1}})\\
&= \sum_{i=1}^{p} \epsilon_{2i-1} w_{f(i)}
\end{aligned}$$

where

$$f(i) = 2i - \theta(\epsilon_{2i-1}).$$

For each $i = 1, \ldots, 2p$, $\overline{w_i} = a^{\sigma_i}$. Let t be the generator of $\pi(L)/Y$ which is identified with $\overline{a} = \overline{b}$ under the quotient map $\pi \circ h$ from (1.1). Up to multiplication by powers of t,

$$\Delta_L(t) = \pi'\Big(\frac{\partial w}{\partial b}\Big) = \sum_{i=1}^{p} \epsilon_{2i-1} t^{\sigma_{f(i)}} \tag{3.3}$$

where $\pi' : \mathbb{Z}[\pi(L)] \to \mathbb{Z}[t]$ is the map induced by $\pi \circ h$.

By Proposition 3.2,

$$R_k = S_{\sigma_{f(1)}}^{\epsilon_1} S_{\sigma_{f(2)}}^{\epsilon_3} \cdots S_{\sigma_{f(p)}}^{\epsilon_{2p-1}}$$

so

$$\begin{aligned} [R_k] =& \epsilon_1 S'_{\sigma_{f(1)}} + \epsilon_3 S'_{\sigma_{f(2)}} + \cdots + \epsilon_{2p-1} S'_{\sigma_{f(p)}} \\ =& \sum_{i=1}^{p} \epsilon_{2i-1} S'_{\sigma_{f(i)}}. \end{aligned} \tag{3.4}$$

Parts (b) and (c) follow from (3.3) and (3.4). □

Chapter 4

Cycle Graphs of Two-Bridge Links

4.1 Cycle Graphs

By Proposition 3.2, the form of the relator R_0 depends on patterns in the sequences of ϵ_i's and σ_i's defined in (3.1) and (3.2). In the spirit of Hirasawa and Murasugi [18], graphs are used in order to gain intuition about how the sequences of ϵ_i's and σ_i's behave; however, the construction here slightly differs from the one Hirasawa and Murasugi used.

4.1.1 Incremental Paths and Cycles

A *graded directed graph* is a directed graph Γ with map $\mathsf{gr} : V(\Gamma) \to \mathbb{Z}$ called the *grading*. Here $V(\Gamma)$ denotes the set of vertices of Γ. Two graded directed graphs Γ and Γ' are *isomorphic* if there is a directed graph isomorphism $f : \Gamma \to \Gamma'$ such that for every vertex P in Γ, $\mathsf{gr}(f(P)) = \mathsf{gr}(P)$. Γ and Γ' are called *relatively isomorphic* if there is a directed graph isomorphism $f : \Gamma \to \Gamma'$ and an integer k such that for every vertex P in Γ, $\mathsf{gr}(f(P)) = \mathsf{gr}(P) + k$.

An *incremental path* is a graded directed path graph Γ where the gradings of adjacent vertices differ by ± 1. Similarly, an *incremental cycle* is a graded directed cycle graph Γ where the gradings of adjacent vertices differ by

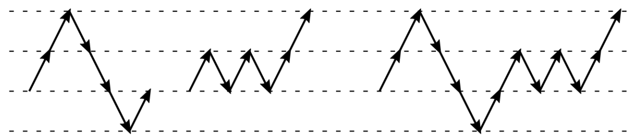


Figure 4.1: The concatenation of Γ and Γ'

± 1. We will at times refer to an incremental path or cycle as an *incremental graph*.

Let Γ and Γ' be two incremental paths in which the grading of the last vertex in Γ is equal to the grading of the first vertex in Γ'. Define the *concatenation* of Γ and Γ', denoted $\Gamma * \Gamma'$, to be the graded directed graph obtained by identifying the last vertex in Γ with the first vertex in Γ' (see Figure 4.1).

If the grading of the first and last vertices in Γ are the same, Γ is called *closable* and the *closure of* Γ, $\mathsf{cl}(\Gamma)$,is defined to be the incremental cycle obtained by identifying the first and last vertex in Γ.

4.1.2 Cycle Graphs of Co-prime Pairs

Let (p, q) denote a co-prime pair of integers p and q such that p is positive, q is odd and $p > |q|$. Define the sequences ϵ_i and σ_i as in (3.1) and (3.2) for each integer i. Define the incremental path $\Gamma(p, q)$ as follows. The vertex set of $\Gamma(p, q)$ is $\{P_0, \ldots, P_{2p}\}$, and the edge set of $\Gamma(p, q)$ is

$$E(\Gamma(p, q)) = \{(P_0, P_1), (P_1, P_2), \ldots, (P_{2p-1}, P_{2p})\}.$$

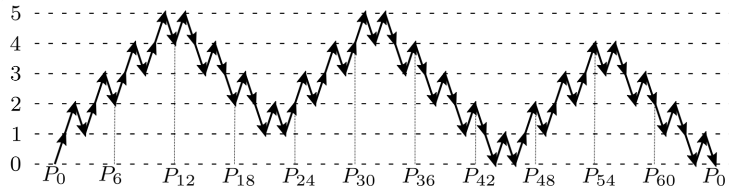


Figure 4.2: $\overline{\Gamma}(33, 23)$

The grading of each vertex is defined by $\mathsf{gr}(P_i) = \sigma_i$. $\Gamma(p, q)$ is always closable, and the *cycle graph of* p *and* q, $\overline{\Gamma}(p, q)$ is defined to be $\mathsf{cl}(\Gamma(p, q))$. When studying $\overline{\Gamma}(p, q)$, it's convenient to think of its vertices $\{P_0, \ldots, P_{2p-1}\}$ being indexed by elements of $\mathbb{Z}/(2p\mathbb{Z})$. See Figure 4.2 for an example.

Proposition 4.1. *Let* (p, q) *be a co-prime pair. The cycle graphs* $\overline{\Gamma}(p, q)$ *and* $\overline{\Gamma}(p, -q)$ *are relatively isomorphic.*

Proof. Let $\{\epsilon_i\}_{i \in \mathbb{Z}}$ be the sequence of signs of (p, q) defined in (3.1). For each integer i, define

$$\varepsilon_i = (-1)^{\lfloor \frac{-iq}{p} \rfloor}$$

which is the sequence of signs of $(p, -q)$. Let q' be the unique integer such that $0 < q' < 2p$ and $q'q \cong p - 1$ modulo $2p$. Then

$$\varepsilon_i = \epsilon_{i+q'} \tag{4.1}$$

for every i in $\mathbb{Z}/(2p\mathbb{Z})$. For each integer $i = 0, \ldots, 2p$, define

$$\varsigma_i := \sum_{j=0}^{i-1} \varepsilon_i,$$

which are the gradings of the vertices of $\overline{\Gamma}(p, -q)$. By (4.1),

$$\varsigma_i = \sigma_{i+q'} - \sigma_{q'}$$

for every positive integer i. Since the σ_i's are the gradings of the vertices of $\overline{\Gamma}(p,q)$, it follows that $\overline{\Gamma}(p,q)$ and $\overline{\Gamma}(p,-q)$ are relatively isomorphic. $\square$

4.1.3 Segments and Blocks

Definition 4.2. Given an incremental cycle Γ, a *positive(negative) k-segment* is a set of k consecutive positive(negative) increment edges in Γ which are followed and preceded by negative(positive) increment edges; see Figure 4.3a.

For each co-prime integer pair (p,q), $\overline{\Gamma}(p,q)$ is the closure of the concatenation of segments of alternating sign as follows.

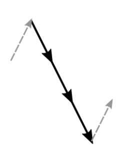
$$\overline{\Gamma}(p,q) = \mathsf{cl}(\Lambda_0 * \Lambda_1 * \cdots * \Lambda_{n-1})$$

As a convention, let Λ_0 denote the segment in $\overline{\Gamma}(p,q)$ containing the edge which corresponds to ϵ_0.

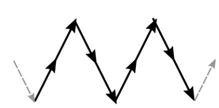
Proposition 4.3 and Proposition 4.5 are analogs of the properties proved in section 6 of Hirasawa and Murasugi's paper [18].

Proposition 4.3. *Let (p,q) be a co-prime pair with $q>0$. Let $P_0,\ldots,P_{2p-1}$ be the vertices of $\overline{\Gamma}(p,q)$ as defined in section 4.1.2, and let*

$$\overline{\Gamma}(p,q) = \mathsf{cl}(\Lambda_0 * \Lambda_1 * \cdots * \Lambda_{n-1})$$



(a) A negative 3-segment



(b) A 2-block of length 4

Figure 4.3

where $\Lambda_0, \ldots, \Lambda_{n-1}$ *are segments. Also, let* κ *and* ξ *be integers such that* $p = \kappa q + \xi$ *and* $0 < \xi < q$.

(a) The number of segments n *in* $\overline{\Gamma}(p, q)$ *is equal to* $2q$.

(b) P_i *is at the beginning of a segment precisely when* $iq \operatorname{\mathsf{mod}} p < q$.

(c) When $\xi \leqslant iq \operatorname{\mathsf{mod}} p < q$, P_i *is at the beginning of a* κ*-segment, and when* $iq \operatorname{\mathsf{mod}} p < \xi$, P_i *is at the beginning of a* $(\kappa + 1)$*-segment.*

(d) Λ_0 *is a* $(\kappa + 1)$*-segment.*

(e) There are a total of 2ξ, $(\kappa + 1)$*-segments in* $\overline{\Gamma}(p, q)$.

Proof. For (a), notice that the segments of $\overline{\Gamma}(p, q)$ correspond to the number of distinct floored quotients $\lfloor \frac{iq}{p} \rfloor$ there are when $i = 0, \ldots, 2p - 1$. Since $p > q$, these quotients range from 0 to $2q - 1$ without skipping so there are exactly $2q$ segments.

A segment begins when

$$\lfloor \frac{(i-1)q}{p} \rfloor \neq \lfloor \frac{iq}{p} \rfloor,$$

which happens when $(iq \operatorname{\mathsf{mod}} p) < q$, proving (b).

For (c), suppose P_i is the beginning of a k-segment. k is the smallest positive integer such that

$$\lfloor \frac{iq}{p} \rfloor \neq \lfloor \frac{(i+k)q}{p} \rfloor.$$

so

$$(iq \operatorname{\mathsf{mod}} p) + (k-1)q < p$$

and

$$(iq \operatorname{\mathsf{mod}} p) + kq \geqslant p.$$

When $\xi \leqslant (iq \operatorname{\mathsf{mod}} p) < q$, $k = \kappa$. Likewise, when $(iq \operatorname{\mathsf{mod}} p) < \xi$, $k = \kappa + 1$.

Parts (d) and (e) immediately follow from (c). □

Definition 4.4. A *k-block of length l* in $\overline{\Gamma}(p, q)$ is a sequence of l consecutive k-segments that is not proceeded or followed by a k-segment; see Figure 4.3b. A k-block of length 1 is called an *isolated block*.

Proposition 4.5. *Let (p, q) be a co-prime pair with $p > q > 0$ and q odd. Let $P_0, \ldots, P_{2p-1}$ be the vertices of $\overline{\Gamma}(p, q)$ as defined in section 4.1.2. Define κ, ξ, κ', and ξ' be integers such that*

$$p = \kappa q + \xi \text{ with } 0 < \xi < q \tag{4.2}$$

and

$$q = \kappa' \xi + \xi' \text{ with } 0 < \xi' < \xi. \tag{4.3}$$

(a) All of the κ-blocks in $\overline{\Gamma}(p, q)$ have length κ' or $\kappa' - 1$.

(b) If P_j is the start of a κ-block, then when

$$q - \xi' \leqslant jq \operatorname{\mathsf{mod}} p < q,$$

the κ-blocks has length κ' and when

$$q - \xi \leqslant jq \bmod p < q - \xi',$$

the κ-blocks has length $\kappa' - 1$.

(c) *If $\kappa' \geqslant 2$ then all the $(\kappa+1)$-blocks in $\overline{\Gamma}(p,q)$ are isolated.*

(d) *If $\kappa' = 1$ then all the κ-blocks in $\overline{\Gamma}(p,q)$ are isolated.*

Proof. Similar to the proof of Proposition 4.3, this proposition is just matter of determining when κ-blocks and $(\kappa+1)$-blocks appear is $\overline{\Gamma}(p,q)$.

Suppose P_i is the beginning of a $(\kappa+1)$-segment. The next segment begins at P_j where $j = i + \kappa + 1$, and by (4.2),

$$\begin{aligned} jq \bmod p =& ((i+\kappa+1)q) \bmod p \\ =& (iq + \kappa q + q) \bmod p \\ =& (iq + p - \xi + q) \bmod p \\ =& ((iq \bmod p) + q - \xi) \bmod p. \end{aligned}$$

Since P_i is the beginning of a $(\kappa+1)$-segment, $(iq \bmod p) < \xi$ by Proposition 4.3(c) so

$$q - \xi \leqslant (iq \bmod p) + q - \xi < q < p. \tag{4.4}$$

Thus,

$$jq \bmod p = (iq \bmod p) + q - \xi. \tag{4.5}$$

For (a) and (b), suppose a κ-block starts at vertex P_j. The length of the κ-block starting at P_j is the smallest positive integer n, such that $P_{s(n)}$ is the start of a $(\kappa+1)$-block where $s(k) = j + k\kappa$ so n is the smallest positive integer such that

$$0 \leqslant s(n)q \,\mathsf{mod}\, p\xi < \xi.$$

By (4.2),

$$\begin{aligned} s(k)q \,\mathsf{mod}\, p =& (j + k\kappa)q \,\mathsf{mod}\, p \\ =& (jq + k\kappa q) \,\mathsf{mod}\, p \\ =& (jq + kp - k\xi) \,\mathsf{mod}\, p \\ =& ((jq \,\mathsf{mod}\, p) - k\xi) \,\mathsf{mod}\, p. \end{aligned}$$

By (4.4) and (4.5), since P_j is the beginning of a κ-segment,

$$q - \xi \leqslant jq \,\mathsf{mod}\, p < q.$$

We compute the length n for each of the two cases $q - \xi \leqslant (jq \,\mathsf{mod}\, p) < q - \xi'$ and $q - \xi' \leqslant (jq \,\mathsf{mod}\, p) < q$.

Suppose that

$$q - \xi' \leqslant jq \,\mathsf{mod}\, p < q. \tag{4.6}$$

By (4.3),

$$((jq \,\mathsf{mod}\, p) - \kappa'\xi = ((jq \,\mathsf{mod}\, p) - q + \xi'$$

and

$$0 \leqslant ((jq \,\mathsf{mod}\, p) - q + \xi' < \xi'$$

so

$$0 \leqslant s(\kappa')q \operatorname{\mathsf{mod}} p < \xi' < \xi.$$

Thus, $n \leqslant \kappa'$.

Suppose $k \leqslant \kappa' - 1$. By (4.3) and (4.6),

$$\begin{aligned}\xi \leqslant &((jq \operatorname{\mathsf{mod}} p) - q + \xi' + \xi \\ =&((jq \operatorname{\mathsf{mod}} p) - \kappa'\xi + \xi \\ =&((jq \operatorname{\mathsf{mod}} p) - (\kappa' - 1)\xi\end{aligned}$$

so

$$\xi \leqslant ((jq \operatorname{\mathsf{mod}} p) - k\xi < q.$$

Thus,

$$\xi \leqslant s(k)q \operatorname{\mathsf{mod}} p < q$$

so $n \geqslant \kappa'$. Therefore, $n = \kappa'$.

Suppose

$$q - \xi \leqslant (jq \operatorname{\mathsf{mod}} p) < q - \xi',$$

By (4.3),

$$((jq \operatorname{\mathsf{mod}} p) - (\kappa' - 1)\xi = ((jq \operatorname{\mathsf{mod}} p) - q + \xi' + \xi$$

and

$$0 \leqslant \xi' \leqslant ((jq \operatorname{\mathsf{mod}} p) - q + \xi' + \xi < \xi$$

so

$$0 \leqslant s(\kappa' - 1)q \operatorname{\mathsf{mod}} p < \xi.$$

Thus, $n \leqslant \kappa' - 1$.

Suppose $k \leqslant \kappa' - 2$. By (4.3) and (4.6),

$$\begin{aligned}\xi \leqslant & ((jq \operatorname{\mathsf{mod}} p) - q + \xi' + 2\xi \\ = & ((jq \operatorname{\mathsf{mod}} p) - (\kappa' - 2)\xi\end{aligned}$$

so

$$\xi \leqslant ((jq \operatorname{\mathsf{mod}} p) - k\xi < q.$$

Thus,

$$\xi \leqslant s(k)q \operatorname{\mathsf{mod}} p < q$$

so $n \geqslant \kappa' - 1$. Therefore, $n = \kappa' - 1$. Thus, all of the κ-blocks have length κ' or $\kappa' - 1$.

For (c), suppose that $\kappa' \geqslant 2$. By (4.3),

$$q - \xi = (\kappa' - 1)\xi + \xi',$$

and since $\kappa' \geqslant 2$,

$$\xi \leqslant \xi + \xi' \leqslant q - \xi$$

so by (4.4),

$$\xi \leqslant (iq \operatorname{\mathsf{mod}} p) + q - \xi < q.$$

Thus, by (4.5),

$$\xi \leqslant jq \operatorname{\mathsf{mod}} p < q.$$

By Proposition 4.3(c), P_j must be the beginning of a κ-segment so $(\kappa + 1)$-segments cannot occur consecutively. Therefore, $(\kappa + 1)$-blocks are isolated.

Statement (d) follows immediately from (a). $\square$

4.2 Reducing and Expanding Cycle Graphs

4.2.1 Reducing Cycle Graphs

Let (p, q) be a co-prime pair with $q > 0$. Let κ, ξ, κ' and ξ' be defined as in Proposition 4.5, and let the decomposition of $\overline{\Gamma}(p, q)$ be

$$\overline{\Gamma}(p, q) = \mathsf{cl}(\Lambda_0 * \cdots * \Lambda_{2q-1}). \tag{4.7}$$

Define a reduction of $\overline{\Gamma}(p, q)$, denoted $R(\overline{\Gamma})(p, q)$, by

1. eliminating all κ-segments,
2. replacing each $(\kappa + 1)$-segment with a positive or negative increment according to the sign of the segment, and
3. setting the grading of the vertex preceding the edge corresponding to Λ_0 equal to zero.

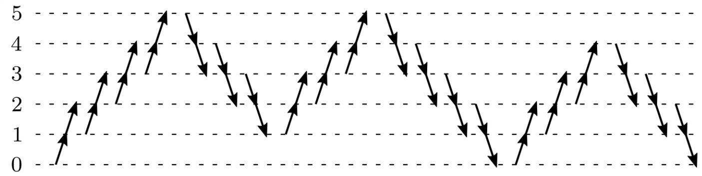
For an example, see Figure 4.4.

Lemma 4.6. *Let (p, q) be a co-prime pair with $q > 1$ and $\xi > 1$. Define p^* to be ξ, and define q^* as follows.*

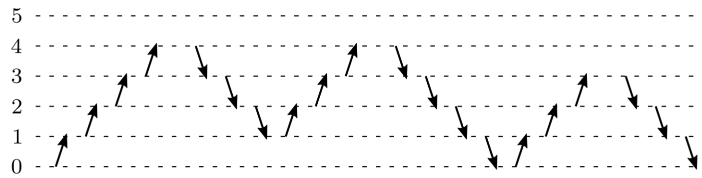
$$q^* = \begin{cases} \xi' & \text{when } \kappa' \text{ is even} \\ \xi' - \xi & \text{when } \kappa' \text{ is odd} \end{cases}$$

(a) p^* is always positive and q^* is always odd.

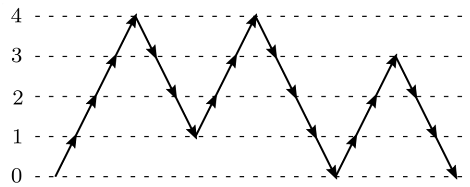
(b) $R(\overline{\Gamma})(p, q)$ is isomorphic to $\overline{\Gamma}(p^*, q^*)$.



(a) All the 1-segments have been removed from $\overline{\Gamma}(33,23)$; see Figure 4.2.



(b) The 2-segments have been replaced by edges.



(c) The resulting graph is $R(\overline{\Gamma})(33,23)$ which is isomorphic to $\overline{\Gamma}(10,3)$.

Figure 4.4: Reducing $\overline{\Gamma}(33,23)$

Proof. For (a), we see that $\xi > 0$ since p and q are co-prime. Also, notice that q is odd and

$$\xi' = q - \kappa'\xi.$$

If κ' is even then $q^* = \xi'$ is odd. If κ' is odd then ξ' and ξ must have opposite parities so $q^* = \xi' - \xi$ is odd.

For (b), consider $\overline{\Gamma}(p,q)$. By Proposition 4.3(e), we know that $\overline{\Gamma}(p,q)$ has 2ξ $(\kappa+1)$-segments so $R(\overline{\Gamma})(p,q)$ has 2ξ edges and 2ξ vertices. Let $\{Q_0,\ldots,Q_{2\xi-1}\}$ be the vertex set of $R(\overline{\Gamma})(p,q)$, and $\{P_0^*,\ldots,P_{2\xi-1}^*\}$ be the vertex set of $\overline{\Gamma}(p^*, q^*)$. Since $R(\overline{\Gamma})(p, q)$ and $\overline{\Gamma}(p^*, q^*)$ are cycle graphs with the same number of vertices, there is a unique ungraded directed graph isomorphism between them by mapping $Q_i \mapsto P_i^*$. Since $\mathsf{gr}(Q_0)$ and $\mathsf{gr}(P_0^*)$ are both 0 by definition, it only remains to show

$$\mathsf{gr}(Q_{i+1}) - \mathsf{gr}(Q_i) = \mathsf{gr}(P_{i+1}^*) - \mathsf{gr}(P_i^*)$$

for each $i = 0, \ldots, 2\xi - 1$.

For $i = 0, \ldots, 2\xi - 1$, define

$$\varepsilon_i := \mathsf{gr}(Q_{i+1}) - \mathsf{gr}(Q_i)$$

and

$$\eta_i := (-1)^{\lfloor \frac{i\xi'}{\xi} \rfloor}.$$

If $q^* = \xi'$, then

$$\mathsf{gr}(P_{i+1}^*) - \mathsf{gr}(P_i^*) = \eta_i,$$

and if $q^* = \xi' - \xi$, then

$$\mathsf{gr}(P_{i+1}^*) - \mathsf{gr}(P_i^*) = (-1)^{\lfloor \frac{i(\xi' - \xi)}{\xi} \rfloor} = (-1)^i \eta_i.$$

Let $j_0, \ldots, j_{2\xi-1}$ be the indices in ascending order of the $(\kappa+1)$-segments in the decomposition in (4.7), and let l_i be the index of the vertex in $\overline{\Gamma}(p, q)$ at the beginning of Λ_{j_i}; see Figure 4.5. By definition of $R(\overline{\Gamma})(p, q)$, ε_i is positive precisely when Λ_{j_i} is a positive segment. Thus, $\varepsilon_{i+1} = \varepsilon_i$ when Λ_{j_i} and $\Lambda_{j_{i+1}}$ are separated by an even number of κ-segments, and $\varepsilon_{i+1} = -\varepsilon_i$ when Λ_{j_i} and

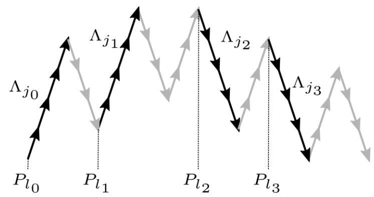


Figure 4.5: The $(\kappa+1)$-segments of $\overline{\Gamma}(17,5)$. The indices of the segments are $j_0 = 0$, $j_1 = 2$, $j_2 = 5$, and $j_3 = 7$. The indices of the vertices at the beginning of each $(\kappa+1)$-segment are $l_0 = 0$, $l_1 = 7$, $l_2 = 17$, and $l_3 = 24$.

$\Lambda_{j_{i+1}}$ are separated by an odd number of κ-segments. The desired result will follow from three claims.

Claim 1: Whenever $0 \leqslant (i\xi' \operatorname{\mathsf{mod}} \xi) < \xi - \xi'$,

$$\eta_{i+1} = \eta_i,$$

and whenever $(i\xi' \operatorname{\mathsf{mod}} \xi) \geqslant \xi - \xi'$,

$$\eta_{i+1} = -\eta_i.$$

When $0 \leqslant (i\xi' \operatorname{\mathsf{mod}} \xi) < \xi - \xi'$, there are integers s and t with

$$i\xi' = s\xi + t \text{ and } 0 \leqslant t < \xi - \xi'$$

so

$$s\xi \leqslant (i+1)\xi' = s\xi + t + \xi' < (s+1)\xi.$$

Thus,

$$\eta_{i+1} = (-1)^s = \eta_i.$$

When $(i\xi' \operatorname{\mathsf{mod}} \xi) \geqslant \xi - \xi'$, there are integers s and t with

$$i\xi' = s\xi + t \text{ and } \xi - \xi' \leqslant t < \xi$$

so

$$(s+1)\xi \leqslant (i+1)\xi' = s\xi + t + \xi' < (s+1)\xi + \xi' < (s+2)\xi.$$

Thus,

$$\eta_{i+1} = (-1)^{s+1} = -\eta_i.$$

Claim 2: The segments Λ_{j_i} *and* $\Lambda_{j_{i+1}}$ *are separated by a* κ*-block of length* κ' *when*

$$\xi - \xi' \leqslant (l_i q \operatorname{\mathsf{mod}} p) < \xi$$

and a κ*-block of length* $\kappa' - 1$ *(possibly zero) when*

$$0 \leqslant (l_i q \operatorname{\mathsf{mod}} p) < \xi - \xi'.$$

By Proposition 4.5(b), every κ-block begins at a vertex P_l where

$$q - \xi \leqslant (lq \operatorname{\mathsf{mod}} p) < q.$$

The length of the block is κ' when

$$q - \xi' \leqslant (lq \operatorname{\mathsf{mod}} p) < q, \tag{4.8}$$

and the length is $\kappa' - 1$ when

$$q - \xi \leqslant (lq \operatorname{\mathsf{mod}} p) < q - \xi'. \tag{4.9}$$

The vertex at the end of the segment Λ_{j_i} is the same as the vertex at the beginning the segment Λ_{j_i+1} so Λ_{j_i+1} begins at the vertex with index $l' := l_i + \kappa + 1$. By Proposition 4.3(b),

$$0 \leqslant l_i q \operatorname{\mathsf{mod}} p + q - \xi < q < p$$

so

$$\begin{aligned} l'q \operatorname{\mathsf{mod}} p =& (l_i + \kappa + 1)q \operatorname{\mathsf{mod}} p \\ =& (l_i q \operatorname{\mathsf{mod}} p + q - \xi) \operatorname{\mathsf{mod}} p \\ =& l_i q \operatorname{\mathsf{mod}} p + q - \xi. \end{aligned}$$

By (4.8), Λ_{j_i} and $\Lambda_{j_{i+1}}$ are separated by a κ-block of length κ' when

$$q - \xi' \leqslant (l'q \operatorname{\mathsf{mod}} p) < q$$

so

$$\xi - \xi' \leqslant (l_i q \operatorname{\mathsf{mod}} p) < \xi.$$

By (4.9), κ-block of length $\kappa' - 1$ when

$$q - \xi \leqslant (l'q \operatorname{\mathsf{mod}} p) < q - \xi'$$

so

$$0 \leqslant (l_i q \operatorname{\mathsf{mod}} p) < \xi - \xi'.$$

Claim 3: For each $i = 0, \ldots, 2\xi - 1$

$$l_i q \operatorname{\mathsf{mod}} p = i\xi' \operatorname{\mathsf{mod}} \xi.$$

P_{l_i} and $P_{l_{i+1}}$ are separated by a $(\kappa+1)$-segment and a κ-block. Therefore, when the length of the κ-block is κ',

$$l_{i+1} = l_i + (\kappa + 1) + \kappa'\kappa$$

so

$$\begin{aligned} l_{i+1}q \operatorname{\mathsf{mod}} p =& (l_iq + \kappa q + q + \kappa'\kappa q) \operatorname{\mathsf{mod}} p \\ =& (l_iq \operatorname{\mathsf{mod}} p + \xi' - \xi) \operatorname{\mathsf{mod}} p \end{aligned}$$

where last equality follows from (4.2) and (4.3). By Claim 2,

$$0 \leqslant l_iq \operatorname{\mathsf{mod}} p + \xi' - \xi < \xi' < p.$$

Therefore,

$$l_{i+1}q \operatorname{\mathsf{mod}} p = l_iq \operatorname{\mathsf{mod}} p + \xi' - \xi. \tag{4.10}$$

When the length of the κ-block is $\kappa' - 1$,

$$l_{i+1} = l_i + (\kappa + 1) + (\kappa' - 1)\kappa = l_i + 1 + \kappa'\kappa$$

so

$$\begin{aligned} l_{i+1}q \operatorname{\mathsf{mod}} p =& (l_iq + q + \kappa'\kappa q) \operatorname{\mathsf{mod}} p \\ =& (l_iq \operatorname{\mathsf{mod}} p + \xi') \operatorname{\mathsf{mod}} p. \end{aligned}$$

By Claim 2,

$$0 < \xi' \leqslant l_iq \operatorname{\mathsf{mod}} p + \xi' < \xi < p.$$

Therefore,

$$l_{i+1}q \operatorname{\mathsf{mod}} p = l_i q \operatorname{\mathsf{mod}} p + \xi'. \tag{4.11}$$

In either the case (4.10) or (4.11),

$$l_{i+1}q \operatorname{\mathsf{mod}} p = (l_i q \operatorname{\mathsf{mod}} p + \xi') \operatorname{\mathsf{mod}} \xi$$

so since $l_0 = 0$,

$$l_i q \operatorname{\mathsf{mod}} p = i\xi' \operatorname{\mathsf{mod}} \xi$$

for each $i = 0, \ldots, 2\xi - 1$ by induction. This completes the proof of the claim.

Suppose κ' is even. When Λ_{i+1} and Λ_i are separated by a κ-block of length $\kappa' - 1$, Λ_{i+1} and Λ_i have the same sign so

$$\varepsilon_{i+1} = \varepsilon_i.$$

By the three claims,

$$0 \leqslant (i\xi' \operatorname{\mathsf{mod}} \xi) < \xi - \xi'$$

so

$$\eta_{i+1} = \eta_i.$$

When Λ_{i+1} and Λ_i are separated by a κ-block of length κ', Λ_{i+1} and Λ_i have opposite signs so

$$\varepsilon_{i+1} = -\varepsilon_i.$$

By the three claims,

$$(i\xi' \operatorname{\mathsf{mod}} \xi) \geqslant \xi - \xi'$$

so

$$\eta_{i+1} = -\eta_i.$$

Since $\varepsilon_0 = \eta_0 = 1$, for every $i = 0, \ldots, 2\xi - 1$,

$$\varepsilon_i = \eta_i$$

so when $q^* = \xi'$,

$$\mathsf{gr}(P^*_{i+1}) - \mathsf{gr}(P^*_i) = \eta_i = \varepsilon_i = \mathsf{gr}(Q_{i+1}) - \mathsf{gr}(Q_i).$$

Suppose κ' is odd. When Λ_{i+1} and Λ_i are separated by a κ-block of length κ', then $\varepsilon_{i+1} = \varepsilon_i$. When Λ_{i+1} and Λ_i are separated by a κ-block of length $\kappa' - 1$, then $\varepsilon_{i+1} = -\varepsilon_i$.

Thus, by the claims, $\varepsilon_{i+1} = \varepsilon_i$ when $\eta_{i+1} = -\eta_i$, and $\varepsilon_{i+1} = -\varepsilon_i$ when $\eta_{i+1} = \eta_i$. Again, $\varepsilon_0 = \eta_0 = 1$. Therefore, for every $i = 0, \ldots, 2\xi - 1$,

$$\varepsilon_i = (-1)^i \eta_i$$

so when $q^* = \xi' - \xi$, then

$$\mathsf{gr}(P^*_{i+1}) - \mathsf{gr}(P^*_i) = (-1)^i \eta_i = \varepsilon_i = \mathsf{gr}(Q_{i+1}) - \mathsf{gr}(Q_i).$$

$\square$

Example 4.7. *Consider the co-prime pair* $(33, 23)$. $R(\overline{\Gamma})(33, 23)$ *is isomorphic to* $\Gamma(10, 3)$ *(see Figure 4.4).*

4.2.2 Expanding Cycle Graphs

We can also reverse the reduction process R. Let Γ be an incremental path with vertices $P_0, \ldots, P_n$ indexed such that (P_i, P_{i+1}) is an edge in Γ for each $i = 0, \ldots, n-1$. Let s and b be positive integers, and let $e = \pm 1$. Define the expansion $\tilde{E}(\Gamma, s, b, e)$ to be the incremental path graph constructed as follows:

1. Create a $(s+1)$-segment, Λ_i, for each edge (P_i, P_{i+1}) in Γ. Choose Λ_i to be positive or negative according to the sign of the edge $(P_i, P_{i+1})'$.

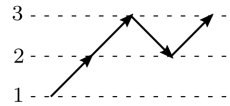
2. Between each pair Λ_i and Λ_{i+1}, for $i = 0, \ldots, n-2$, add a s-block of length b or $b-1$. The length of the s-block is odd if the edges Λ_i and Λ_{i+1} have the same sign, and the length is even if Λ_i and Λ_{i+1} have opposite signs. Also, the first s-segment in the block has sign opposite of the sign of Λ_i.

3. Add another s-block to the beginning of Λ_i of length b or $b-1$ depending on the signs of Λ_0 and e following the same convention as the previous step. Also, the first s-segment in the block has sign opposite of e.

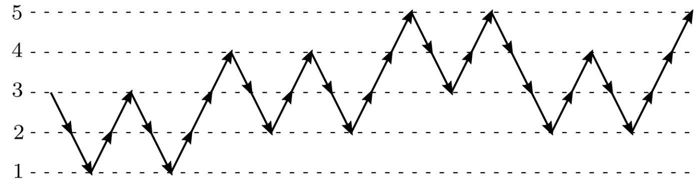
4. Finally, set the grading of the first vertex Q_0 as follows.

$$\mathsf{gr}(Q_0) = \begin{cases} \mathsf{gr}(P_0) + s & \text{when } e \text{ and } (P_0, P_1) \text{ are both positive} \\ \mathsf{gr}(P_0) - s & \text{when } e \text{ and } (P_0, P_1) \text{ are both negative} \\ \mathsf{gr}(P_0) & \text{when } e \text{ and } (P_0, P_1) \text{ have opposite sign} \end{cases} \tag{4.12}$$

For an example, see Figure 4.6.



(a) An incremental path Γ



(b) $\tilde{E}(\Gamma, 2, 3, +)$

Figure 4.6

By construction, the following property holds

Lemma 4.8. *Suppose Γ and Γ' are isomorphic incremental paths. For any positive integers s and b and any sign $e = \pm 1$,*

$$\tilde{E}(\Gamma, s, b, e) \cong \tilde{E}(\Gamma', s, b, e).$$

We begin by investigating the gradings of the vertices in $\tilde{E}(\Gamma, s, b, e)$. Let Q_0 be the vertex at the beginning of $\tilde{E}(\Gamma, s, b, e)$. For $i = 1, \ldots, n$, let Q_i be the vertex at the end of $(s+1)$-segment Λ_{i-1} as defined in the definition of $\tilde{E}$.

Lemma 4.9. *For each $i = 1, \ldots, n$,*

(a) if the sign of Λ_i and e are the same, then

$$\mathsf{gr}(Q_i) - \mathsf{gr}(Q_0) = \mathsf{gr}(P_i) - \mathsf{gr}(P_0),$$

(b) if Λ_{i-1} is positive and e is negative, then

$$\mathsf{gr}(Q_i) - \mathsf{gr}(Q_0) = \mathsf{gr}(P_i) - \mathsf{gr}(P_0) + s, \textit{and}$$

(c) if Λ_{i-1} is negative and e is positive, then

$$\mathsf{gr}(Q_i) - \mathsf{gr}(Q_0) = \mathsf{gr}(P_i) - \mathsf{gr}(P_0) - s.$$

Proof. The vertices Q_0 and Q_i are separated some number of segments. Let D^+ and D^- be the number of positive or negative $(s+1)$-segments. Likewise, let d^+ and d^- be the number of positive or negative s-segments. Note that D^+ and D^- are also the number of positive and negative edges in Γ so

$$D^+ - D^- = \mathsf{gr}(P_i) - \mathsf{gr}(P_0).$$

Suppose Λ_{i-1} and e are have the same sign, then the number of positive segments in $\tilde{E}(\Gamma, s, b, e)$ is equal to the number of negative segments so

$$D^+ + d^+ = D^- + d^-.$$

Thus,

$$\begin{aligned}
\mathsf{gr}(Q_i) - \mathsf{gr}(Q_0) =& D^+(s+1) - D^-(s+1) + d^+ s - d^- s \\
=& (D^+ + d^+)s - (D^- + d^-)s + D^+ - D^- \\
=& D^+ - D^- \\
=& \mathsf{gr}(P_i) - \mathsf{gr}(P_0).
\end{aligned}$$

Suppose Λ_{i-1} is positive and e is negative, then the total number of positive segments in $\tilde{E}(\Gamma, s, b, e)$ is one more than the total number of negative segments so

$$\begin{aligned}
\mathsf{gr}(Q_i) - \mathsf{gr}(Q_0) =& D^+(s+1) - D^-(s+1) + d^+s - d^-s \\
=& (D^+ + d^+)s - (D^- + d^-)s + D^+ - D^- \\
=& s + D^+ - D^- \\
=& \mathsf{gr}(P_i) - \mathsf{gr}(P_0) + s.
\end{aligned}$$

Suppose Λ_{i-1} is negative and e is positive, then the total number of positive segments in $\tilde{E}(\Gamma, s, b, e)$ is one less than the total number of negative segments so

$$\begin{aligned}
\mathsf{gr}(Q_i) - \mathsf{gr}(Q_0) =& D^+(s+1) - D^-(s+1) + d^+s - d^-s \\
=& (D^+ + d^+)s - (D^- + d^-)s + D^+ - D^- \\
=& -s + D^+ - D^- \\
=& \mathsf{gr}(P_i) - \mathsf{gr}(P_0) - s.
\end{aligned}$$

□

From this, we can show that concatenation behaves well under expansion.

Lemma 4.10. *Suppose Γ and Γ' are incremental paths where the last vertex in Γ has the same grading as the first vertex in Γ'. Let e' be the sign of the last edge in Γ. For any positive integers s and b and any sign $e = \pm 1$,*

$$\tilde{E}(\Gamma * \Gamma', s, b, e) \cong \tilde{E}(\Gamma, s, b, e) * \tilde{E}(\Gamma', s, b, e').$$

Proof. The conclusion will be true by definition of the expansion procedure as long as $\tilde{E}(\Gamma, s, b, e)$ and $\tilde{E}(\Gamma', s, b, e')$ can be concatenated. Thus, our goal is to show that the last vertex in $\tilde{E}(\Gamma, s, b, e)$ has the grading as the first vertex in $\tilde{E}(\Gamma', s, b, e')$. This can be done by computing $\tilde{E}(\Gamma * \Gamma', s, b, e)$ for many cases depending on the signs of e, the last edge in Γ, and the first edge in Γ'.

For example, suppose e, the last edge in Γ, and the first edge in Γ' are all positive. Let P_0 and P_n be the first and last vertices of Γ. Let P_0' be the first vertex in Γ' so $\mathsf{gr}(P_n) = \mathsf{gr}(P_0')$. Let Q_0 and Q_n be the first and last vertices of $\tilde{E}(\Gamma, s, b, e)$. Finally, let Q_0' be the first vertex in $\tilde{E}(\Gamma', s, b, e')$.

By (4.12),

$$\mathsf{gr}(Q_0') = \mathsf{gr}(P_0') + s = \mathsf{gr}(P_n) + s$$

By Lemma 4.9,

$$\begin{aligned}\mathsf{gr}(Q_n) =& \mathsf{gr}(P_n) - \mathsf{gr}(P_0) + \mathsf{gr}(Q_0)\\ =& \mathsf{gr}(Q_0') - s - \mathsf{gr}(P_0) + \mathsf{gr}(P_0) + s\\ =& \mathsf{gr}(Q_0').\end{aligned}$$

The proofs of all the other cases are similar. □

Let Γ be a closable incremental path, and let e be the sign of the last edge in Γ. For any two positive integers s and b, define

$$E(\Gamma, s, b) := \tilde{E}(\Gamma, s, b, e).$$

When Γ is closable, $E(\Gamma, s, b)$ is also closable.

Suppose Γ' is a closable incremental path such that $\mathsf{cl}(\Gamma) \cong \mathsf{cl}(\Gamma')$. By construction,

$$\mathsf{cl}(E(\Gamma, s, b)) \cong \mathsf{cl}(E(\Gamma', s, b)) \tag{4.13}$$

for all positive integers s and b.

For a incremental cycle $\overline{\Gamma}$, define

$$E(\overline{\Gamma}, s, b) := \mathsf{cl}(E(\Gamma, s, b)).$$

where Γ is any incremental path such that $\mathsf{cl}(\Gamma) \cong \overline{\Gamma}$. By (4.13), $E(\overline{\Gamma}, s, b)$ is well-defined.

By construction reduction and expansion natural opposite operations.

Proposition 4.11. *Suppose (p, q) is a co-prime pair with $q > 0$. Define κ and κ' as in (4.2) and (4.3).*

$$E(R(\overline{\Gamma})(p, q), \kappa, \kappa') \cong \overline{\Gamma}(p, q)$$

Given an arbitrary co-prime pair (p^*, q^*) and integers s and b, $E(\overline{\Gamma}(p^*, q^*), s, b)$ may not be $\overline{\Gamma}(p, q)$ for any co-prime (p, q) with q odd. Consider the pair $(5, 3)$. Suppose $E(\overline{\Gamma}(5, 3), 2, 3) \cong \overline{\Gamma}(p, q)$ for some pair (p, q). Then, $q = 3(5) + 3 = 18$.

Chapter 5

Residual Torsion-Free Nilpotence, Bi-Orderablity and Two-Bridge Links

5.1 The Alexander Subgroup and Baumslag's Conditions

By Proposition 2.7 and Theorem 1.9, Theorem A follows from the following lemma.

Lemma 5.1. *Suppose L is an oriented two-bridge link whose Alexander polynomial has relatively prime coefficients. The Alexander subgroup Y of L can be written as a union of an ascending chain of subgroups $Y_0 < Y_1 < Y_2 < \cdots < Y$ such that*

(a) each Y_n is parafree of the same rank and

(b) $|Y_{n+1} : Y_n[Y_{n+1}, Y_{n+1}]|$ is finite for each n.

Theorem A. *Let L be an oriented two-bridge link. If the Alexander polynomial of L has relatively prime coefficients and all real and positive roots, then $\pi(L)$ is bi-orderable. In particular, if J is a two-bridge knot and all the roots of the Alexander polynomial of J are real and positive, then the knot group of J is bi-orderable.*

Proof. Suppose L is an oriented two-bridge link whose Alexander polynomial has relatively prime coefficients and all real positive roots. By Lemma 5.1 and Proposition 2.7, the Alexander subgroup of L is residually torsion free nilpotent. Therefore, by Theorem 1.9, $\pi(L)$ is bi-orderable.

Since the Alexander polynomials of knots always have relatively prime coefficients, the Alexander subgroups of two-bridge knots are always residually torsion-free nilpotent. Therefore, when a knot's Alexander polynomial has all real positive roots, its knot group is bi-orderable. □

Thus, our goal now is to show that every two-bridge link whose Alexander polynomial has relatively prime coefficients satisfies Lemma 5.1. In a talk, Mayland [34] proposes a strategy that to describe the commutator subgroup of a two-bridge knot group as the union of an ascending chain of subgroups satisfying the conditions of Lemma 5.1. The first term Y_0 is a free group, and ideally, for each $n \geqslant 1$, Y_n is isomorphic to Y_{n-1} after adjoining roots of homologically primitive elements, in the manner of Proposition 2.10, a finite number of times. While it is straightforward to verify Mayland's argument on a case by case basis, proving his strategy works in general is quite difficult.

Here we use reductions and expansions of cycle graphs to relate the Alexander subgroups of more complicated two-bridge link groups to those of simpler two-bridge link groups. Then, it is proven inductively that the Alexander subgroups of all two-bridge links can be described by adjoining roots to a free group, and we show that when two-bridge links have Alexander polynomials with relatively prime coefficients, their Alexander subgroups satisfy Lemma 5.1 via Mayland's strategy.

5.1.1 An Example

Here, we use the two-bridge knot $K := L(17/13)$ to provide an example of the proof of Lemma 5.1. Using the Schubert normal form [48], we obtain a presentation of $\pi(K)$.

$$\pi(K) = \langle a, b \mid avb^{-1}v^{-1} \rangle$$

where

$$v = ba^{-1}ba^{-1}b^{-1}ab^{-1}aba^{-1}ba^{-1}b^{-1}ab^{-1}a.$$

Denote the Alexander subgroup of $\pi(K)$ by Y. Using the Reidemeister-Schreier rewriting process, we obtain the following presentation of Y.

$$Y \cong \langle \{S_k\}_{k\in\mathbb{Z}} \mid \{R_k\}_{k\in\mathbb{Z}} \rangle$$

where $S_k = a^k ba^{-k-1}$ and the relators R_k are defined as follows.

$$R_k = S_{k+1}^2 S_k^{-2} S_{k+1}^2 S_k^{-3} S_{k-1}^2 S_k^{-2} S_{k-1}^2 S_k^{-2}$$

Define a sequence of groups $\{Y_n\}_{n=0}^{\infty}$ as follows.

$$\begin{aligned}
Y_0 :=& \langle S_{-1}, S_0 \rangle \\
Y_1 :=& \langle S_{-2}, S_{-1}, S_0, S_1 \mid R_{-1}, R_0 \rangle \\
Y_2 :=& \langle S_{-3}, S_{-2}, S_{-1}, S_0, S_1, S_2 \mid R_{-2}, R_{-1}, R_0, R_1 \rangle \\
&\vdots
\end{aligned}$$

Define $\widehat{A}_1$, $\widehat{A}_2$, $\widehat{V}_1$ and $\widehat{V}_2$ as follows.

$$\begin{aligned}
\widehat{A}_1 &= S_1^2 S_0^{-2} \\
\widehat{A}_2 &= S_1 \\
\widehat{V}_1 &= S_0^{-1} S_{-1}^2 S_0^{-2} S_{-1}^2 S_0^{-2} \\
\widehat{V}_2 &= S_0^{-2}
\end{aligned} \tag{5.1}$$

Let H_1 be the group obtained by adjoining a square root of $\widehat{V}_1^{-1}$ to Y_0 as follows.

$$H_1 := Y_0 \underset{\widehat{V}_1^{-1} = t_1^2}{*} \langle t_1 \rangle$$

Similarly, let H_2 be the group obtained by adjoining a square root of $t_1 \widehat{V}_2^{-1}$ to H_1.

$$H_2 := Y_1 \underset{t_1 \widehat{V}_2^{-1} = S_1^2}{*} \langle S_1 \rangle$$

Thus, H_2 has the following group presentation.

$$\begin{aligned}
H_2 &\cong \langle S_{-1}, S_0, S_1, t_1 \mid t_1^2 \widehat{V}_1 = 1, t_1 = S_1^2 \widehat{V}_2 \rangle \\
&\cong \langle S_{-1}, S_0, S_1 \mid (S_1^2 \widehat{V}_2)^2 \widehat{V}_1 = 1, \rangle \\
&\cong \langle S_{-1}, S_0, S_1 \mid R_0 \rangle
\end{aligned}$$

Define $\check{A}_1$, $\check{A}_2$, $\check{V}_1$ and $\check{V}_2$ as follows.

$$\begin{aligned}
\check{A}_1 &= S_{-2}^2 S_{-1}^{-2} \\
\check{A}_2 &= S_{-2} \\
\check{V}_1 &= S_0^2 S_{-1}^{-2} S_0^2 S_{-1}^{-3} \\
\check{V}_2 &= S_{-1}^{-2}
\end{aligned} \tag{5.2}$$

Let H_3 be the group obtained by adjoining a square root of $\check{V}_1^{-1}$ to H_2.

$$H_3 := H_2 \underset{\check{V}_1^{-1}=t_2^2}{*} \langle t_2 \rangle$$

Let H_4 be the group obtained by adjoining a square root of $t_2\check{V}_2^{-1}$ to H_3.

$$H_4 := H_3 \underset{t_2\check{V}_2^{-1}=S_{-2}^2}{*} \langle S_{-2} \rangle$$

Therefore, H_4 is isomorphic to Y_1.

$$\begin{aligned} H_4 \cong & \langle S_{-2}, S_{-1}, S_0, S_1, t_2 \mid \check{V}_1 t_2^2 = 1, t_2 = S_{-2}^2 \check{V}_2 \rangle \\ \cong & \langle S_{-2}, S_{-1}, S_0, S_1 \mid R_{-1}, R_0 \rangle \\ \cong & Y_1 \end{aligned}$$

In conclusion, Y_1 is Y_0 after adjoining roots four times, and since $R_{n\pm 1}$ is R_n with all the subscripts changed by ± 1, Y_{n+1} is Y_n after adjoining roots four times. Thus, for each n, Y_n embeds into Y_{n+1}, and $|Y_{n+1} : Y_n[Y_{n+1}, Y_{n+1}]|$ is finite. Therefore, Y is the union of an ascending chain of subgroups as follows.

$$Y_0 < Y_1 < \cdots < Y = \bigcup_{n=0}^{\infty} Y_n$$

By Proposition 2.7, if each Y_n is parafree of the same rank then Y is residually torsion-free nilpotent. Y_0 is clearly parafree of rank 2 since it is a rank 2 free group. We need to verify that each time we adjoin a root of an element, that element is homologically primitive. Then, by Proposition 2.10, we can conclude that each Y_n is also parafree of rank 2.

Claim: For each $n \geqslant 0$, if Y_n is parafree of rank 2, then so is Y_{n+1}.

Proof. Let n be a non-negative integer, and suppose Y_n is parafree of rank 2. In an abuse of notation, let $\widehat{A}_1$, $\widehat{A}_2$, $\widehat{V}_1$ and $\widehat{V}_2$ be as defined in (5.1) except with the subscripts of each S_i increased by n. Similarly, let $\check{A}_1$, $\check{A}_2$, $\check{V}_1$ and $\check{V}_2$ be as defined in (5.2) except with the subscripts of each S_i decreased by n. Also, let H_1, H_2, H_3 and H_4 be the groups obtained by adjoining square roots of $\widehat{V}_1^{-1}$, $t_1\widehat{V}_2^{-1}$, $\check{V}_1^{-1}$ and $t_2\check{V}_2^{-1}$ to Y_n as before.

Let Y_n^{ab} denote the abelianization of Y_n, and let B_1 be the quotient of Y_n^{ab} obtained by killing the class of $\widehat{V}_1^{-1}$ in Y_n^{ab}. Since Y_n is parafree of rank 2, $Y_n^{\mathsf{ab}} \cong \mathbb{Z} \oplus \mathbb{Z}$. Thus,

$$B_1 \cong \mathbb{Z} \oplus \frac{\mathbb{Z}}{C\mathbb{Z}}$$

for some integer C.

Now, we view Y_n^{ab} as a $\mathbb{Z}$-module and use addition as the group operation. Y_n^{ab} is generated by $S'_{-n-1}, S'_{-n}, \ldots, S'_n$ where S'_i donotes the class of S_i in Y_n^{ab}. Using this generating set, Y_n^{ab} has a $(2n) \times (2n+2)$ presentation matrix:

$$\begin{pmatrix} 4 & -9 & 4 & & & \\ & 4 & -9 & 4 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 4 & -9 & 4 \end{pmatrix}.$$

The class of $\widehat{V}_1^{-1}$ in Y_n^{ab} is $-4S'_{n-1} + 5S'_n$. Thus, B_1 has the following $(2n+1) \times (2n+2)$ presentation matrix, which we will also call B_1.

$$B_1 = \begin{pmatrix} 4 & -9 & 4 & & & \\ & 4 & -9 & 4 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 4 & -9 & 4 \\ & & & & -4 & 5 \end{pmatrix}$$

By Lemma 2.1, the integer C is the greatest common divisor of the determinants of every $(2n+1)\times(2n+1)$ submatrix of B_1. By deleting the last column, we get a square submatrix of B_1 with determinant -4^{2n+1}. However, by deleting the first column, we see B_1 has a submatrix with odd determinant. (Modulo 2, B_1 is the identity matrix.) Thus, $C = 1$.

Therefore, B_1 is a rank 1 free abelian group. It follows that $\widehat{V}_1^{-1}$ is homologically primitive in Y_n, and H_1 is parafree of rank 2 by Proposition 2.10.

Let B_2 be the quotient of H_1^{ab} obtained by killing the class of $t_1\widehat{V}_2^{-1}$ in H_1^{ab}, the abelianization of H_1. H_1^{ab} is generated by $S'_{-n-1}, S'_{-n}, \ldots, S'_n, t'_1$ where t'_1 is the class of t_1 in H_1^{ab}. H_1^{ab} has a $(2n+1)\times(2n+3)$ presentation matrix:

$$\begin{pmatrix} 4 & -9 & 4 & & & & \\ & 4 & -9 & 4 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 4 & -9 & 4 & \\ & & & & -4 & 5 & 2 \end{pmatrix}.$$

The class of $t_1\widehat{V}_2^{-1}$ in H_1^{ab} is $2S'_n + t'_1$. Thus, B_2 has the following $(2n+2)\times(2n+3)$ presentation matrix.

$$B_2 = \begin{pmatrix} 4 & -9 & 4 & & & & \\ & 4 & -9 & 4 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 4 & -9 & 4 & \\ & & & & 4 & -5 & 2 \\ & & & & & 2 & 1 \end{pmatrix}$$

Using the 1 in the bottom right corner, we apply a row operation and kill the last row and column to get the following presentation matrix.

$$B_2 \cong \begin{pmatrix} 4 & -9 & 4 & & & & \\ & 4 & -9 & 4 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 4 & -9 & 4 \\ & & & & 4 & -9 \end{pmatrix}$$

Thus, B_2 is a rank 1 free abelian group, by a argument similar to the one used for B_1. It follows that $t_1\widehat{V}_2^{-1}$ is homologically primitive in H_1, and H_2 is parafree of rank 2 by Proposition 2.10.

Similarly, $\check{V}_1^{-1}$ and $t_2\check{V}_2^{-1}$ are homologically primitive in H_2 and H_3 respectively. Therefore, $H_4 \cong Y_{n+1}$ is parafree of rank 2. $\square$

Since Y_0 is parafree of rank 2, each Y_n is parafree of rank 2 by induction. Also, $|Y_{n+1} : Y_n[Y_{n+1}, Y_{n+1}]| = 16$. Therefore, Y is residually torsion-free nilpotent by Proposition 2.7.

5.1.2 An Ascending Chain of Subgroups

In chapter 3, we found a group presentation of the Alexander subgroup of an arbitrary two-bridge link group using the Reidemeister-Schreier rewriting process. From this presentation of the Alexander subgroup, we can describe the subgroup as the union of an ascending chain of subgroups which satisfy conditions (a) and (b) of Lemma 5.1 when the Alexander polynomial of the link has relatively prime coefficients.

Define Y_0 to be the free group

$$Y_0 := \langle S_m, S_{m+1}, \ldots, S_{M-1} \rangle, \tag{5.3}$$

and define Y_n to be the group with presentation

$$Y_n := \langle S_{m-n}, S_{m-n+1}, \ldots, S_{M+n-1} \mid R_{-n}, \ldots, R_{n-1} \rangle. \tag{5.4}$$

for each positive integer n.

Y_{n+1} is Y_n with two extra generators, S_{m-n-1} and S_{M+n}, and two extra relators, R_{-n-1} and R_n. It turns out that all of the appearances of S_{M+n} in R_n are contained in nested repeating patterns of words. Similarly, all of the appearances of S_{m-n-1} in R_{-n-1} are contained in nested repeating patterns of words. Given an explicit two-bridge link, one can find these patterns easily, as we did in section 5.1.1 for $L(17/13)$, yet showing that these patterns exist for any two-bridge knot is much more complicated.

Once it is established that these patterns exists, however, it follows that for each non-negative integer n, Y_{n+1} is Y_n after adjoining roots a finite number of times. This implies that each Y_n embeds into Y_{n+1}. Since Y is the direct limit of the sequence of Y_n's, Y is the union of the ascending chain of Y_n's. When the coefficients of Δ_L are relatively prime, the elements whose roots are adjoining are homologically primitive.

The following lemma explicitly describes the relator R_0 (and hence any R_n by Proposition 3.2) as nested patterns of repeating words.

Lemma 5.2. *There exist a positive integer N, sequences of words in $\mathcal{S}$,*

$$\widehat{A}_0, \widehat{A}_1, \ldots, \widehat{A}_N,$$

and

$$\widehat{V}_1, \dots, \widehat{V}_N,$$

and a sequence of positive integers $n_1, \dots, n_N$ *such that all of the following hold:*

(M1) $R_0 = \widehat{A}_0$,

(M2) $\widehat{A}_N = S_M^{\pm 1}$,

(M3) for each $i = 1, \dots, N$, $\widehat{A}_{i-1} = \widehat{A}_i^{n_i} \widehat{V}_i$ *(up to conjugation),*

(M4) for each $i = 1, \dots, N$, $S_M^{\pm 1}$ *does not appear in* $\widehat{V}_i$, *and*

(M5) for each $i = 1, \dots, N$, *there is some* l *with* $m < l \leqslant M$ *and integers* $b_l, \dots, b_M$ *(which depend on* i*) such that*

$$[\widehat{A}_i] = \sum_{j=l}^{M} b_j S'_j = b_l S'_l + b_{l+1} S'_{l+1} + \cdots + b_M S'_M$$

with $|b_{l+j}| = |b_{M-j}|$.

Also, there are sequences

$$\check{A}_0, \check{A}_1, \dots, \check{A}_N,$$

and

$$\check{V}_1, \dots, \check{V}_N,$$

such that

(m1) $R_0 = \check{A}_0$,

(m2) $\breve{A}_N = S_m^{\pm 1}$,

(m3) for each $i = 1, \ldots, N$, $\breve{A}_{i-1} = \breve{A}_i^{n_i} \breve{V}_i$ *(up to conjugation),*

(m4) for each $i = 1, \ldots, N$, $S_m^{\pm 1}$ *does not appear in* $\breve{V}_i$, *and*

(m5) for each $i = 1, \ldots, N$, *there is some* l' *with* $m \leqslant l' < M$, *and integers* $b_m, \ldots, b_{l'}$ *(which depend on* i*) such that*

$$[\breve{A}_i] = \sum_{j=m}^{l'} b_j S'_j = b_m S'_m + \cdots + b_{l'} S'_{l'}$$

with $|b_{m+j}| = |b_{l'-j}|$.

Remark 5.1.1. Y_1 is obtained from Y_0 by adding $2N$ roots. In order of increasing index, each $\widehat{A}_i$ is added as the n_ith root of some element, then each $\breve{A}_i$ is added as an n_ith root. The conditions (M5) and (m5) are used to show that the elements whose roots are added are homologically primitive.

Lemma 5.2 is reinterpreted in terms of cycle graphs in section 5.2 and proven in section 5.3.

Proposition 5.3. *The Alexander subgroup* Y *of any oriented two-bridge link is a union of an ascending chain of subgroups*

$$Y_0 < Y_1 < Y_2 < \cdots < Y_i < \cdots < \bigcup_{n=1}^{\infty} Y_n \cong Y$$

where Y_{n+1} *is obtained from* Y_n *by adjoining a finite number of roots.*

Proof. Define the sequence $Y_0, Y_1, Y_2, \ldots$ as in (5.3) and (5.4). Consider Y_n for some non-negative integer n.

$$Y_n = \langle S_{m-n}, \ldots, S_{M+n-1} \mid R_{-n}, \ldots, R_{n-1} \rangle$$

and

$$Y_{n+1} = \langle S_{m-n-1}, \ldots, S_{M+n} \mid R_{-n-1}, \ldots, R_n \rangle.$$

By Proposition 3.2(d) and Lemma 5.2 there is an integer N, sequences of words

$$\widehat{A}_0, \ldots, \widehat{A}_N,$$

and

$$\widehat{V}_1, \ldots, \widehat{V}_N,$$

and a sequence of integers

$$n_1, \ldots, n_N.$$

such that

$$\widehat{A}_0 = R_n,$$

$$\widehat{A}_N = S^{\pm}_{M+n},$$

and for some $\widehat{W}_i$,

$$\widehat{W}_i^{-1} \widehat{A}_{i-1} \widehat{W}_i = \widehat{A}_i^{n_i} \widehat{V}_i$$

for each $i = 1, \ldots, N$.

Let $\langle t_i \rangle$ be an infinite cyclic group generated by t_i for each $i = 1, \ldots, N$. Also, let t_0 be trivial in Y_n.

Define

$$H_0 = Y_n, \tag{5.5}$$

and for each $i = 1, \ldots, N$, recursively define

$$H_i = H_{i-1} \underset{\widehat{h}_i = t_i^{n_i}}{*} \langle t_i \rangle \tag{5.6}$$

where

$$\widehat{h}_i = \widehat{W}_i^{-1} t_{i-1} \widehat{W}_i \widehat{V}_i^{-1}.$$

Thus,

$$\begin{aligned} H_N \cong \langle S_{m-n}, \ldots, S_{M+n}, t_1, \ldots, t_N \mid R_{-n}, \ldots, R_{n-1}, \\ \{\widehat{h}_i^{-1} t_i^{n_i}\}_{i=2}^{N}, \\ \widehat{V}_1 t_1^{n_1}, t_N^{-1} \widehat{A}_N \rangle. \end{aligned}$$

By backwards substitution using (M1), (M2), and (M3) of Lemma 5.2,

$$\begin{aligned} H_N \cong & \langle S_{m-n}, \ldots, S_{M+n}, t_1, \ldots, t_N \mid R_{-n}, \ldots, R_{n-1}, \widehat{A}_0, t_1^{-1} \widehat{A}_1, \ldots, t_N^{-1} \widehat{A}_N \rangle \\ \cong & \langle S_{m-n}, \ldots, S_{M+n} \mid R_{-n}, \ldots, R_n \rangle. \end{aligned}$$

Likewise, by Proposition 3.2(d) and Lemma 5.2 there are sequences of words

$$\check{A}_0, \ldots, \check{A}_N,$$

and

$$\check{V}_1, \ldots, \check{V}_N,$$

such that

$$\check{A}_0 = R_{-n-1},$$

$$\breve{A}_N = S^{\pm}_{m-n-1},$$

and for some $\breve{W}_i$,

$$\breve{W}_i^{-1}\breve{A}_{i-1}\breve{W}_i = (\breve{A}_i)^{n_i}\breve{V}_i$$

for each $i = 1, \ldots, N$.

For each $i = 1, \ldots, N$, define

$$H_{i+N} = H_{i+N-1} \underset{\breve{h}_i = t_i^{n_i}}{*} \langle t_i \rangle \tag{5.7}$$

where

$$\breve{h}_i = \breve{W}_i^{-1} t_{i-1} \breve{W}_i \breve{V}_i^{-1}.$$

$$\begin{aligned} H_{2N} \cong \langle S_{m-n-1}, \ldots, S_{M+n}, t_1, \ldots, t_N \mid R_{-n}, \ldots, R_n, \\ \{\breve{h}_i^{-1} t_i^{n_i}\}_{i=2}^{N}, \\ \breve{V}_1 t_1^{n_1}, t_N^{-1}\breve{A}_N \rangle. \end{aligned}$$

By backwards substitution using (m1), (m2), and (m3) of Lemma 5.2,

$$\begin{aligned} H_{2N} \cong & \langle S_{m-n-1}, \ldots, S_{M+n}, t_1, \ldots, t_N \mid R_{-n}, \ldots, R_{n-1}, \\ & \qquad \breve{A}_0, t_1^{-1}\breve{A}_1, \ldots, t_N^{-1}\breve{A}_N \rangle \\ \cong & \langle S_{m-n-1}, \ldots, S_{M+n} \mid R_{-(n+1)}, \ldots, R_n \rangle \\ \cong & Y_{n+1}. \end{aligned} \tag{5.8}$$

Consider Y_n and Y_{n+1} for a non-negative integer n. For each $i = 0, \ldots, 2N-1$, H_i embeds into H_{i+1} since H_{i+1} is a free product of H_i and

$\mathbb{Z}$ amalgamated along infinite cyclic subgroups. Let $\varphi_i : H_i \to H_{i+1}$ be the embedding which maps $S_k \mapsto S_k$ and $t_k \mapsto t_k$ for all k. The composition $f_n = \varphi_{2N-1} \circ \cdots \circ \varphi_0$ is an embedding of Y_n into Y_{n+1} which maps $S_k \mapsto S_k$ for all k.

Thus, we have the following sequence of embeddings.

$$Y_0 \xhookrightarrow{f_0} Y_1 \xhookrightarrow{f_1} Y_2 \xhookrightarrow{f_2} \cdots \xhookrightarrow{f_{n-1}} Y_n \xhookrightarrow{f_n} \cdots$$

The Alexander subgroup Y is the direct limit of this sequence, since each f_n is an embedding, Y is a union of an ascending chain of subgroups as desired. $\square$

5.1.3 Proof of Lemma 5.1

We now turn our attention to proving Lemma 5.1. First, we state a more precise and detailed version of Lemma 5.1.

Lemma 5.4. *Suppose that Y is the Alexander subgroup of a two-bridge link whose Alexander polynomial has relatively prime coefficients so that Y is an ascending chain of subgroups*

$$Y_0 < Y_1 < Y_2 < \cdots < Y = \bigcup_{n=1}^{\infty} Y_n$$

as defined in (5.3) and (5.4). For each n,

(a) Y_n is parafree of the rank $M - m$ and

(b) $|Y_{n+1} : Y_n[Y_{n+1}, Y_{n+1}]| = \underline{a}_g^2$ where $\underline{a}_g$ is the leading coefficient of the Alexander polynomial of L.

Proof. First we show (a). Y_0 is a parafree of rank $M-m$ since it's a rank $M-m$ free group. Suppose that for some $n \geqslant 0$, Y_n is parafree of rank $M-m$. Define $H_0, \dots, H_{2N}$ as in (5.5), (5.6), and (5.7) so $H_{2N} \cong Y_{n+1}$ as in (5.8).

Suppose H_{k-1} is parafree of rank $M-m$ for some k such that $0 < k \leqslant N$ so $H_{k-1}^{\mathsf{ab}} \cong \mathbb{Z}^{M-m}$. Define

$$B := \frac{H_{k-1}}{\langle \widehat{h}_k \rangle [H_{k-1}, H_{k-1}]} \cong \mathbb{Z}^{M-m-1} \oplus \frac{\mathbb{Z}}{C\mathbb{Z}}$$

where

$$\widehat{h}_k = \widehat{W}_k^{-1} t_{k-1} \widehat{W}_k \widehat{V}_k^{-1}$$

and C is an integer. If $B \cong \mathbb{Z}^{M-m-1}$, then $\widehat{h}_k$ is homologically primitive in H_{k-1}, and inductively, by Proposition 2.10, each H_k is parafree of rank $M-m$.

By Proposition 3.4, $H_0^{\mathsf{ab}} = Y_n^{\mathsf{ab}}$ has $2n \times 2n + M - m$ presentation matrix

$$\begin{pmatrix} a_m & a_{m+1} & \cdots & a_{M-1} & a_M & & & \\ & \ddots & & \ddots & & \ddots & & \\ & & & a_m & a_{m+1} & \cdots & a_{M-1} & a_M \end{pmatrix}.$$

H_{k-1} is H_0 with the n_j root of $\widehat{h}_j$ added for each $j = 1, \dots, k-1$. Thus, B is H_0^{ab} after killing the classes $[\widehat{h}_j^{-1} t_j^{n_j}]$ for each $j = 1, \dots, k-1$. B is generated by $S'_{m-n}, \dots, S'_{M+n-1}, t'_1, \dots, t'_{k-1}$ where t'_j is the class $[t_j]$. Using these generators,

B has the following $(2n+k)\times(2n+k+M-m-1)$ presentation matrix.

$$\begin{pmatrix}
a_m & a_{m+1} & \cdots & a_{M-1} & a_M & & & & & & \\
& \ddots & & \ddots & & \ddots & & & & & \\
& & a_m & a_{m+1} & \cdots & a_{M-1} & a_M & & & & \\
& & 0 & \longleftarrow & [\widehat{V}_1] & \longrightarrow & n_1 & & & & \\
& & 0 & \longleftarrow & [\widehat{V}_2] & \longrightarrow & -1 & n_2 & & & \\
& & 0 & \longleftarrow & [\widehat{V}_3] & \longrightarrow & 0 & -1 & n_3 & & \\
& & & & \vdots & & & & \ddots & \ddots & \\
& & 0 & \longleftarrow & [\widehat{V}_{k-1}] & \longrightarrow & 0 & \cdots & 0 & -1 & n_{k-1} \\
& & 0 & \longleftarrow & [\widehat{V}_k] & \longrightarrow & 0 & & \cdots & 0 & -1
\end{pmatrix}.$$

Applying the row operations $\mathbf{row}_j + n_{j+1}\mathbf{row}_{j+1} \to \mathbf{row}_j$ for each row $j = k-1,\ldots,1$ results in the matrix

$$\begin{pmatrix}
a_m & a_{m+1} & \cdots & a_{M-1} & M_g & & & & & & \\
& \ddots & & \ddots & & \ddots & & & & & \\
& & a_m & a_{m+1} & \cdots & a_{M-1} & a_M & & & & \\
& & 0 & \longleftarrow & [U_1] & \longrightarrow & 0 & & & & \\
& & 0 & \longleftarrow & [U_2] & \longrightarrow & -1 & 0 & & & \\
& & 0 & \longleftarrow & [U_3] & \longrightarrow & 0 & -1 & 0 & & \\
& & & & \vdots & & & & \ddots & \ddots & \\
& & 0 & \longleftarrow & [U_{k-1}] & \longrightarrow & 0 & \cdots & 0 & -1 & 0 \\
& & 0 & \longleftarrow & [U_k] & \longrightarrow & 0 & & \cdots & 0 & -1
\end{pmatrix}$$

where

$$[U_j] = [\widehat{V}_j] + n_1([\widehat{V}_{j+1}] + n_2([\widehat{V}_{j+2}] + \cdots + n_{k-2}([\widehat{V}_{k-1}] + n_{k-1}[\widehat{V}_k])\cdots)).$$

Eliminating the last $k-1$ rows and columns results in $(2n+1)\times(2n+M-m)$ the presentation matrix D

$$D = \begin{pmatrix}
a_m & a_{m+1} & \cdots & a_{M-1} & a_M & & & & \\
& a_m & a_{m+1} & \cdots & a_{M-1} & a_M & & & \\
& & \ddots & & \ddots & & \ddots & & \\
& & & a_m & a_{m+1} & \cdots & a_{M-1} & a_M & \\
& & & & c_m & c_{m+1} & \cdots & c_{M-1} &
\end{pmatrix}$$

where

$$[U_1] = c_m S'_{m+n} + c_{m+1} S'_{m+n+1} + \cdots + c_{M-1} S'_{M+n-1}.$$

By Lemma 5.2(M5), for some l with $m < l \leqslant M$, there are integers $b_l, \ldots, b_M$ such that

$$[\widehat{A}_k] = \sum_{j=l}^{M} b_j S'_{j+n} \tag{5.9}$$

and $|b_{l+j}| = |b_{M-j}|$.

Claim 1: For each $j = m, \ldots, M-1$,

$$c_j = \begin{cases} a_j & \textit{when } m \leqslant j < l \\ a_j - (\prod_{s=1}^{k} n_s) b_j & \textit{when } l \leqslant j < M-1 \end{cases}.$$

From the row operations,

$$\begin{aligned}
[U_1] =& [\widehat{V}_1] + n_1([\widehat{V}_2] + n_2([\widehat{V}_3] + \cdots + n_{k-2}([\widehat{V}_{k-1}] + n_{k-1}[\widehat{V}_k]) \cdots)) \\
=& [\widehat{V}_1] + n_1[\widehat{V}_2] + n_1 n_2[\widehat{V}_3] + \cdots + (\prod_{s=1}^{k-2} n_s)[\widehat{V}_{k-1}] + (\prod_{s=1}^{k-1} n_s)[\widehat{V}_k] \\
=& \sum_{j=1}^{k} (\prod_{s=1}^{j-1} n_s)[\widehat{V}_j].
\end{aligned}$$

By Lemma 5.2(M3), $\widehat{V}_j = \widehat{A}_j^{-n_j} \widehat{W}_j^{-1} \widehat{A}_{j-1} \widehat{W}_j$ so $[\widehat{V}_j] = [\widehat{A}_{j-1}] - n_j[\widehat{A}_j]$. Thus,

$$\begin{aligned}
\sum_{j=1}^{k} (\prod_{s=1}^{j-1} n_s)[\widehat{V}_j] &= \sum_{j=1}^{k} (\prod_{s=1}^{j-1} n_s)([\widehat{A}_{j-1}] - n_j[\widehat{A}_j]) \\
&= \sum_{j=1}^{k} (\prod_{s=1}^{j-1} n_s)[\widehat{A}_{j-1}] - \sum_{j=1}^{k} (\prod_{s=1}^{j} n_s)[\widehat{A}_j] \\
&= [\widehat{A}_0] - (\prod_{s=1}^{k} n_s)[\widehat{A}_k].
\end{aligned}$$

Therefore, since $\widehat{A}_0 = R_n$,

$$[U_1] = [R_n] - (\prod_{s=1}^{k} n_s)[\widehat{A}_k]. \tag{5.10}$$

The statement of the claim follows from Proposition 3.4(a), (5.9), and (5.10).

By Lemma 2.1, C is the gcd of all the $(2n+1) \times (2n+1)$ minors of D. Suppose a prime d divides C so d divides the determinant of every $(2n+1) \times (2n+1)$ submatrix of D. The determinant of the submatrix of D given by the first $2n+1$ columns is $-a_m^{2n+1}$ so d divides a_m.

Claim 2: There is some $(2n+1) \times (2n+1)$ submatrix of D whose determinant is, not divisible by d.

By Proposition 3.4(c), the integers $a_m, \ldots, a_M$ are the coeffiecients of the Alexander polynomial. Since the coefficients of $\Delta_L(t)$ are relatively prime, there is some coefficient that d does not divide. Let $m+i$ be the minimal index such that d does not divide a_{m+i}. We prove this claim in two cases.

Case 1. Suppose $m+i < l$, d divides some n_s with $s \leqslant k$, or d divides b_j for all $j = l, \ldots, i$. Then, either $m+i < l$ or d must divide $(\prod_{s=1}^{k} n_s)b_j$ for all $j = l, \ldots, m+i$. By Claim 1, d divides c_j when $j < m+i$ and d doesn't divide c_{m+i}.

Let E be the $(2n+1) \times (2n+1)$ submatrix of D consisting of the $n+1$ consecutive columns starting with the first row which with a_{m+i} (or c_{m+i} if

$n = 0$) at the top. Thus, working modulo d, we have the following submatrix.

$$E = \begin{pmatrix} a_{m+i} & * & * & \cdots & * & * \\ 0 & a_{m+i} & * & \cdots & * & * \\ 0 & 0 & a_{m+i} & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{m+i} & * \\ 0 & 0 & 0 & \cdots & 0 & c_{m+i} \end{pmatrix}.$$

Since d doesn't divide a_{m+i} or c_{m+i}, d cannot divide $\det(E)$.

Case 2. Suppose that $l \leqslant m + i$, d does not divide any n_s with $s \leqslant k$, and there is some $j \leqslant m + i$ such that d does not divide b_j.

Let F_1 be the $(2n + 1) \times 2n$ submatrix given by the n consecutive columns with the coefficient a_{M-i}. By Proposition 3.4(c), $a_{m+j} = a_{M-j}$ for all $j = 0, M - m$ so $M - i$ is the maximal index such that d divides a_{M-i}. Thus, modulo d, F_1 has the following form.

$$F_1 = \begin{pmatrix} a_{M-i} & 0 & 0 & \cdots & 0 \\ * & a_{M-i} & 0 & \cdots & 0 \\ * & * & a_{M-i} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & a_{M-i} \\ * & * & * & \cdots & * \end{pmatrix}.$$

We need to find a column in D with the first n entries divisible by d and the last entry not divisible by d.

Let $l + i'$ be the minimal index such that d does not divide $b_{l+i'}$ so $l + i' \leqslant m + i$.

Since d does not divide $b_{l+i'}$ and $b_{l+i'} = b_{M-i'}$, d does not divide $b_{M-i'}$. By Lemma 5.2(M4), for all j, the coefficient of S'_{M+n} in $[\widehat{V}_j]$ is zero so by (5.10),

$$a_M = b_M \prod_{s=1}^{k} n_s.$$

Since $a_m = a_M$ and d divides a_m, d must also divide b_M. Therefore, d divides b_l so $i' > 0$ and $M - i' \leqslant M - 1$.

Since $M - i' \leqslant M - 1$ there is some column F_2 which ends with $c_{M-i'}$. Every other entry in F_2 is 0 or a_j for some $j > M - i'$. Since $l + i' \leqslant m + i$ and $m < l$,

$$0 < l - m \leqslant i - i'$$

so $M - i < M - i'$. Thus, by Claim 1, d does not divide $c_{M-i'}$, and for all $j > M - i'$, d divides a_j.

Combine F_1 and F_2 to get an $(2n+1) \times (2n+1)$ submatrix F of D. Working modulo d, we have the submatrix.

$$F = \begin{pmatrix} a_{M-i} & 0 & 0 & \cdots & 0 & 0 \\ * & a_{M-i} & 0 & \cdots & 0 & 0 \\ * & * & a_{M-i} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & a_{M-i} & 0 \\ * & * & * & \cdots & * & c_{M-i'} \end{pmatrix}.$$

Since d doesn't divide a_{M-i} or $c_{M-i'}$, d cannot divide $\det(F)$.

In conclusion, there are no primes which divide every determinant of $(2n+1) \times (2n+1)$ submatrices of D so $C = 1$. Thus, $B \cong \mathbb{Z}^{M-m-1}$, and H_k is parafree of rank $M - m$. By induction, H_N is parafree of rank $M - m$.

By a similar induction argument, $H_N, \ldots, H_{2N}$ are also parafree of rank $M-m$. Therefore, $Y_{n+1} \cong H_{2N}$ is parafree of rank $M-m$ so by induction Y_n is parafree of rank $M-m$ for each non-negative integer n.

For (b), consider the group $Y_{n+1}/Y_n[Y_{n+1}, Y_{n+1}]$ which is an abelian group with the following presentation.

$$\frac{Y_{n+1}}{Y_n[Y_{n+1}, Y_{n+1}]} \cong \langle S'_{m-n-1}, \ldots, S'_{M+n} \mid [R_{-n-1}], \ldots, [R_n], S'_{m-n}, \ldots, S'_{M+n-1} \rangle$$

By Proposition 3.4,

$$[R_j] = \underline{a}_g S'_{M+j} + \underline{a}_{g-1} S'_{M-1+j} + \cdots + \underline{a}_{g-1} S'_{m+1+j} + \underline{a}_g S'_{m+j}.$$

After eliminating the generators $S'_{m-n}, \ldots, S'_{M+n-1}$, we have that

$$\frac{Y_{n+1}}{Y_n[Y_{n+1}, Y_{n+1}]} \cong \langle S'_{m-n-1}, S'_{M+n} \mid \underline{a}_g S'_{M-n-1}, \underline{a}_g S'_{m+n} \rangle$$

so

$$\left| Y_{n+1}/Y_n[Y_{n+1}, Y_{n+1}] \right| = \left| \frac{\mathbb{Z}}{\underline{a}_g \mathbb{Z}} \oplus \frac{\mathbb{Z}}{\underline{a}_g \mathbb{Z}} \right| = \underline{a}_g^2.$$

□

5.2 Lemma 5.2 and Cycle Graphs

In this section, we reinterpret Lemma 5.2 as set of properties of the cycle graph $\overline{\Gamma}(p,q)$. These properties will hold for simple co-prime pairs (p,q) with $q = 1$ or $p \,\mathsf{mod}\, q = 1$. Then, it is shown these conditions hold for any co-prime pair of integers p and q with p positive and q odd by a strong induction argument using the relative isomorphism between $\overline{\Gamma}(p,q)$ and $\overline{\Gamma}(p,-q)$ and the reduction from $\overline{\Gamma}(p,q)$ to $R(\overline{\Gamma})(p,q)$.

5.2.1 Making Words From Graphs

Given an incremental path Γ, a word $\rho(\Gamma)$ in $\mathcal{S}$ can be defined as follows. Let $\{P_1, \ldots, P_n\}$ be the vertices of Γ indexed so that the edge (P_i, P_{i+1}) is in Γ. For $i = 2, \ldots, n$, let $s_i = \mathsf{gr}(P_i) - \mathsf{gr}(P_{i-1})$ and let $N_i = \mathsf{gr}(Q_i) + \theta(s_i)$ where $\theta(1) = 1$ and $\theta(-1) = 0$. Define

$$\rho(\Gamma) := \begin{cases} S_{N_3}^{s_3} S_{N_5}^{s_5} \cdots S_{N_k}^{s_k} & \text{if } n > 2 \text{ and } \mathsf{gr}(P_1) \text{ is even} \\ S_{N_2}^{s_2} S_{N_4}^{s_4} \cdots S_{N_k}^{s_k} & \text{if } n > 1 \text{ and } \mathsf{gr}(P_1) \text{ is odd} \\ 1 & \text{otherwise} \end{cases} \tag{5.11}$$

where $k = n - 1$ if $n \equiv \mathsf{gr}(P_1)$ modulo 2, and $k = n$ if $n \not\equiv \mathsf{gr}(P_1)$ modulo 2. Given a two-bridge link $L(p/q)$, by Proposition 3.2, $\rho(\Gamma(p, q))$ is the word R_0.

Lemma 5.5. *Given incremental paths Γ and Γ' such that the last vertex of Γ has the same grading as the first vertex of Γ',*

$$\rho(\Gamma * \Gamma') = \rho(\Gamma)\rho(\Gamma').$$

Proof. Let $\{P_1, \ldots, P_n\}$ and $\{P'_1, \ldots, P'_{n'}\}$ be the vertex sets for incremental paths Γ and Γ' respectively. Also, define $N_2, \ldots, N_n$ and $s_2, \ldots, s_n$ for Γ as in the definition of ρ. Similarly, define $N'_2, \ldots, N'_{n'}$ and $s'_2, \ldots, s'_{n'}$ for Γ'. Let $\Gamma'' = \Gamma * \Gamma'$, which has length $n + n' - 1$, and define $N''_2, \ldots, N''_{n+n'-1}$ and $s''_2, \ldots, s''_{n+n'-1}$ for Γ'' as the analogous integers are defined for Γ and Γ'.

This result is just a matter of computing $\rho(\Gamma * \Gamma')$ for each case of (5.11) for Γ and Γ'. For example, suppose $\mathsf{gr}(P_1)$ and n are even, $n > 2$, and $n' > 1$. Then, since n is even,

$$\mathsf{gr}(P'_1) = \mathsf{gr}(P_n) \equiv (\mathsf{gr}(P_1) + n - 1) \equiv \mathsf{gr}(P_1) + 1 \pmod 2$$

so since $\mathsf{gr}(P_1)$ is even, $\mathsf{gr}(P_1')$ is odd. Thus,

$$\rho(\Gamma) = S_{N_3}^{s_3} S_{N_5}^{s_5} \cdots S_{N_{n-1}}^{s_{n-1}}$$

and

$$\rho(\Gamma') = S_{N_2}^{s_2} S_{N_4}^{s_4} \cdots S_{N_k}^{s_k}$$

where $k = n'$ when n' is even and $k = n' - 1$ when n' is odd.

For each $i = 1, \ldots, n + n' - 1$,

$$\mathsf{gr}(P_i'') = \begin{cases} \mathsf{gr}(P_i) & \text{when } 1 \leqslant i \leqslant n \\ \mathsf{gr}(P_{i-n+1}') & \text{when } n \leqslant i \leqslant n + n' - 1 \end{cases}.$$

Thus, when $2 \leqslant i \leqslant n$, $s_i'' = s_i$ and $N_i'' = N_i$, and when $n + 1 \leqslant i \leqslant n + n' - 1$, $s_i'' = s_{i-n+1}$ and $N_i'' = N_{i-n+1}$. Therefore,

$$\rho(\Gamma * \Gamma') = S_{N_3}^{s_3} S_{N_5}^{s_5} \cdots S_{N_{n-1}}^{s_{n-1}} S_{N_2'}^{s_2'} S_{N_4'}^{s_4'} \cdots S_{N_k'}^{s_k'} = \rho(\Gamma)\rho(\Gamma')$$

The proofs of all the other cases are similar. □

Lemma 5.6. *Given two closable incremental paths Γ and Γ' such that $\mathsf{cl}(\Gamma)$ is isomorphic to $\mathsf{cl}(\Gamma')$, there is a subgraph Υ of Γ such that*

$$\rho(\Gamma') = \rho(\Upsilon)^{-1}\rho(\Gamma)\rho(\Upsilon).$$

Proof. If $\mathsf{cl}(\Gamma) \cong \mathsf{cl}(\Gamma')$ then there are some graphs Υ and Ω such that $\Gamma = \Upsilon * \Omega$ and $\Gamma' = \Omega * \Upsilon$ (see Figure 5.1 for an example). Therefore,

$$\rho(\Gamma') = \rho(\Omega)\rho(\Upsilon) = \rho(\Upsilon)^{-1}\rho(\Upsilon)\rho(\Omega)\rho(\Upsilon) = \rho(\Upsilon)^{-1}\rho(\Gamma)\rho(\Upsilon)$$

□

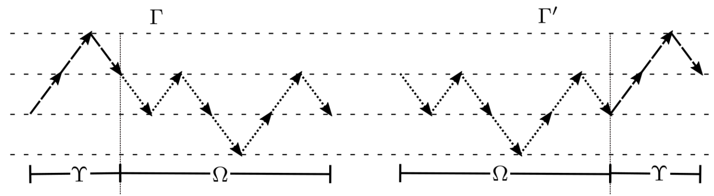


Figure 5.1: Closable graphs Γ and Γ' with isomorphic closures with the subgraphs Υ (dashed) and Ω (dotted) shown.

5.2.2 Summits and Bottoms in Cycle Graphs

Let (p, q) be a co-prime pair, and define M and m for $L(p/q)$ as in chapter 3. In Lamma 5.2, we are primarily interest in the appearances of $S_M^\pm$ and $S_m^\pm$ in the word R_0. When M is odd, the ith S-generator of R_0 is $S_M^\pm$ precisely when $\sigma_{2i} = M + 1$, and when M is even, the ith S-generator of R_0 is $S_M^\pm$ when $\sigma_{2i-1} = M + 1$. Thus, appearances $S_M^\pm$ in R_0 correspond to the indices when σ_i is maximal. Similarly, the ith S-generator of R_0 is $S_m^\pm$ precisely when $\sigma_{2i-1} = m$ when m is odd or $\sigma_{2i} = m$ when m is even. Thus, appearances $S_m^\pm$ in R_0 correspond to the indices when σ_i is minimal.

Definition 5.7. A vertex, P, in a graded graph Γ is called a *summit* if $\mathsf{gr}(P) \geqslant \mathsf{gr}(Q)$ for any vertex Q in Γ. Similarly, P is called a *bottom* if $\mathsf{gr}(P) \leqslant \mathsf{gr}(Q)$ for any vertex Q in Γ.

For each co-prime pair (p, q) the grading of a summit of $\Gamma(p, q)$ is always $M + 1$ and the grading of a bottom of $\Gamma(p, q)$ is always m. Furthermore, the appearances of S_M in R_0 correspond precisely to the summits in $\Gamma(p, q)$, and the appearances of S_m correspond to bottoms.

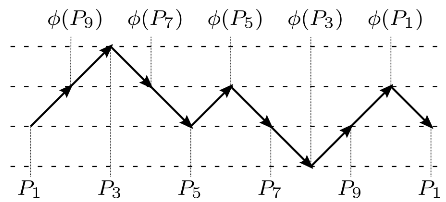


Figure 5.2: A symmetric incremental cycle. The first and last vertices are identified. ϕ is the unique order reversing bijection defined by $\phi(P_1) = P_{10}$.

5.2.3 Symmetric Incremental Paths and Cycles

It is useful to know when an incremental cycle is relatively isomorphic to itself after rotating 180° and reversing its edges. More precsiely, we call an incremental cycle Γ *symmetric* if there is a bijection $\phi : V(\Gamma) \to V(\Gamma)$ such that

1. (P, Q) is an edge of Γ if and only if $(\phi(Q), \phi(P))$ is an edge of Γ for any two vertices P and Q in Γ and
2. for some integer k, $\mathsf{gr}(P) + \mathsf{gr}(\phi(P)) = k$ for every vertex P in Γ.

An incremental path Γ is called *symmetric* if $\mathsf{cl}(\Gamma)$ is symmetric (see Figure 5.2). The symmetry of incremental paths and cycles plays an important role in investigating properties (M5) and (m5) of Lemma 5.2.

5.2.4 Reinterpretation of Lemma 5.2

Here we reinterpret Lemma 5.2 in terms of incremental paths and cycles. Given a closable incremental path Γ and positive integer n, define Γ^n to be the concatenation of n copies of Γ.

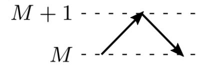


Figure 5.3: The graph Γ_{top}

Definition 5.8. We call a co-prime pair (p, q) an *pre-RTFN pair* if there is a positive integer N, sequences of incremental paths

$$\Gamma_0, \ldots, \Gamma_N$$

and

$$\Upsilon_1, \ldots, \Upsilon_N$$

and a sequence of positive integers

$$n_1, \ldots, n_N$$

such that the following conditions are satisfied:

(R1) $\Gamma_0 = \Gamma(p, q)$,

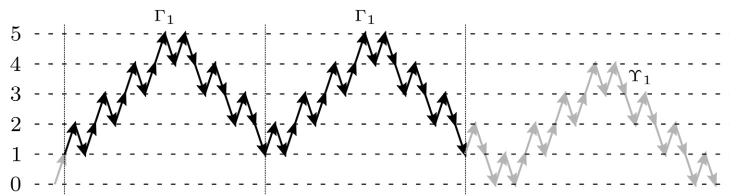
(R2) Γ_N is isomorpic to the graph Γ_{top} defined in Figure 5.3.

(R3) for each $i = 1, \ldots, N$,

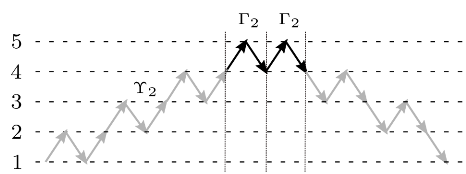
$$\mathsf{cl}(\Gamma_{i-1}) \cong \mathsf{cl}(\Gamma_i^{n_i} * \Upsilon_i),$$

(R4) for each $i = 1, \ldots, N$, no summits appear in Υ_i, and

(R5) for each $i = 0, \ldots, N$, Γ_i is symmetric, and when $i \geqslant 1$, Γ_i contains no bottoms.



(a) $\Gamma_0 = \overline{\Gamma}(33, 23)$ with Υ_1 in gray



(b) Γ_1 with Υ_2 in gray

Figure 5.4: $(33, 23)$ is a pre-RTFN pair.

For an example, Figure 5.4 demonstrates that $(33, 23)$ is a pre-RTFN pair.

Lemma 5.9. *(p, q) is a pre-RTFN pair if and only if $(p, -q)$ is a pre-RTFN pair.*

Proof. This follows immediately from Proposition 4.1. $\square$

Lemma 5.10. *Suppose (p, q) is a co-prime pair. If (p, q) is a pre-RTFN pair, then $L(p/q)$ satisfies Lemma 5.2.*

Proof. Let (p, q) be a pre-RNTF pair. For each $i = 0, \ldots, N$, define

$$\widehat{A}_i := \rho(\Gamma_i),$$

and when $i > 0$, define

$$\widehat{V}_i := \rho(\Upsilon_{N-i}).$$

Proof of (M1) and (M2). By (R1) and (R2),

$$\widehat{A}_0 = \rho(\Gamma_0) = \rho(\Gamma(p, q)) = R_0,$$

and

$$A_N = \rho(\Gamma_N) = S_M^{\pm 1}.$$

Proof of (M3). Suppose i is an integer with $1 \leqslant i \leqslant N$. By (R3),

$$\mathsf{cl}(\Gamma_{i-1}) \cong \mathsf{cl}(\Gamma_i^{n_i} * \Upsilon_i)$$

so by Lemma 5.6, there exists a word W such that

$$\rho(\Gamma_{i-1}) = W^{-1}\rho(\Gamma_i^{n_i} * \Upsilon_i)W.$$

Therefore,

$$\begin{aligned}\widehat{A}_{i-1} =& \rho(\Gamma_{i-1}) \\ =& W^{-1}\rho(\Gamma_i^{n_i} * \Upsilon_i)W \\ =& W^{-1}\widehat{A}_i^{n_i}\widehat{V}_i W.\end{aligned}$$

Proof of (M4). For each $i = 1, \ldots, N$, since no summits appear in Υ_i, $S_M^{\pm 1}$ cannot appear in $\widehat{V}_i$.

Proof of (M5). Suppose i is an integer with $0 \leqslant i \leqslant N$. The maximum grading of a vertex in Γ_i is $M + 1$. Let l be the minimum grading of a vertex in Γ_i. For some integer coefficients $b_l, b_{l+1} \ldots, b_M$,

$$[\rho(\Gamma_i)] = b_l S'_l + b_{l+1} S'_{l+1} + \cdots + b_M S'_M.$$

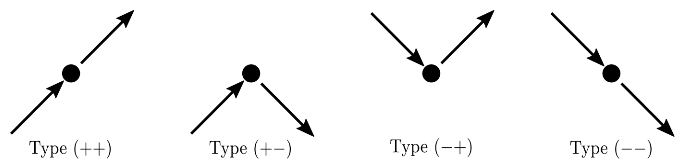


Figure 5.5: The four vertex types

Our goal is to show that for each $j = 0, \ldots, M - l$, $|b_{l+j}| = |b_{M-j}|$.

The vertices of $\mathsf{cl}(\Gamma_i)$ can be classified into four types according to Figure 5.5. Define $v_{(**)}(n)$ to be the number vertices in $\mathsf{cl}(\Gamma_i)$ of type $(**)$ with grading n.

Suppose $n = l, \ldots, M$. When n is even, S_n always has exponent -1 in $\rho(\Gamma_i)$, and S_n^{-1} appears precisely when there is negative edge followed a vertex in $\mathsf{cl}(\Gamma_i)$ with grading n so

$$|b_n| = v_{(--)}(n) + v_{(-+)}(n). \tag{5.12}$$

Similarly, When n is odd, S_n always has exponent 1 in $\rho(\Gamma_i)$, and S_n appears precisely when there is a vertex in $\mathsf{cl}(\Gamma_i)$ with grading n followed by a positive edge so

$$|b_n| = v_{(++)}(n+1) + v_{(+-)}(n+1). \tag{5.13}$$

Since Γ_i is symmetric by (R5), there is an order reversing bijection ϕ of the vertex set of $\mathsf{cl}(\Gamma_i)$ such that $\mathsf{gr}(P) + \mathsf{gr}(\phi(P)) = l + M + 1$ for each vertex P in $\mathsf{cl}(\Gamma_i)$. Furthermore, P and $\phi(P)$ have types rotated 180° with arrows

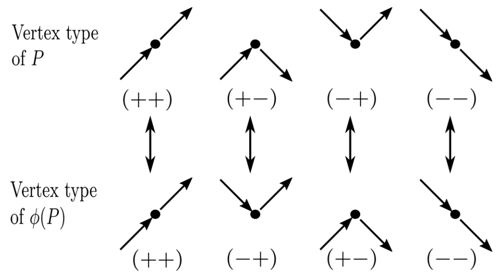


Figure 5.6: The effect of ϕ on vertex type

reversed (see Figure 5.6). As a consequence,

$$\begin{aligned} v_{(--)}(n) &= v_{(--)}(l+M+1-n) \\ v_{(-+)}(n) &= v_{(+-)}(l+M+1-n) \\ v_{(++)}(n) &= v_{(++)}(l+M+1-n) \\ v_{(+-)}(n) &= v_{(-+)}(l+M+1-n) \end{aligned}. \tag{5.14}$$

Each positive edge connects a vertex of type $(*+)$ to a vertex of type $(+*)$. Likewise, each negative edge connects a vertex of type $(*-)$ to a vertex of type $(-*)$ (see Figure 5.7). Thus,

$$\begin{aligned} v_{(++)}(n) + v_{(-+)}(n) &= v_{(++)}(n+1) + v_{(+-)}(n+1) \\ v_{(--)}(n) + v_{(+-)}(n) &= v_{(--)}(n-1) + v_{(-+)}(n-1) \end{aligned}. \tag{5.15}$$

Since Γ_i is closable and the gradings of adjacent vertices differ by ± 1, every time Γ_i passes from below to above some grading level at a vertex, Γ_i must

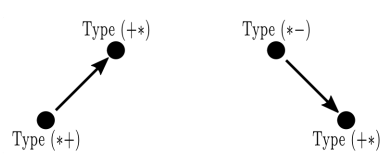


Figure 5.7: Tpye of vertices that are adjacent

pass from above to below the same grading level at some other vertex. Thus, in each grading n,

$$v_{(++)}(n) = v_{(--)}(n). \tag{5.16}$$

Now, we show that $|b_{l+j}| = |b_{M-j}|$. Let j be an integer such that $0 \leqslant j \leqslant M-l$. When $l+j$ and $M-j$ are both even, by (5.12), (5.14), and (5.15),

$$\begin{aligned}
|b_{l+j}| =& v_{(--)}(l+j) + v_{(-+)}(l+j) \\
=& v_{(--)}(M-j+1) + v_{(+-)}(M-j+1) \\
=& v_{(--)}(M-j) + v_{(-+)}(M-j) \\
=& |b_{M-j}|.
\end{aligned}$$

When $l+j$ and $M-j$ are odd, by (5.13), (5.14), and (5.15)

$$\begin{aligned}
|b_{l+j}| =& v_{(++)}(l+j+1) + v_{(+-)}(l+j+1) \\
=& v_{(++)}(M-j) + v_{(-+)}(M-j) \\
=& v_{(++)}(M-j+1) + v_{(+-)}(M-j+1) \\
=& |b_{M-j}|.
\end{aligned}$$

When $l+j$ is even and $M-j$ is odd, by (5.12), (5.14), (5.16), and

(5.13),

$$\begin{aligned}|b_{l+j}| =&v_{(--)}(l+j) + v_{(-+)}(l+j)\\ =&v_{(--)}(M-j+1) + v_{(+-)}(M-j+1)\\ =&v_{(++)}(M-j+1) + v_{(+-)}(M-j+1)\\ =&|b_{M-j}|.\end{aligned}$$

When $l+j$ is odd and $M-j$ is even, by (5.13), (5.14), (5.16), and (5.12),

$$\begin{aligned}|b_{l+j}| =&v_{(++)}(l+j+1) + v_{(+-)}(l+j+1)\\ =&v_{(++)}(M-j) + v_{(-+)}(M-j)\\ =&v_{(--)}(M-j) + v_{(-+)}(M-j)\\ =&|b_{M-j}|.\end{aligned}$$

When $i \geqslant 1$, no bottoms appear in Γ_i so $l > m$.

Proof of (m1), (m2), (m3), (m4), and (m5). Since $\Gamma_0 = \Gamma(p,q)$ is symmetric, there is an order reversing bijection $\overline{\phi}$ on the vertices of $\overline{\Gamma}$ such that

$$\mathsf{gr}(P) + \mathsf{gr}(\overline{\phi}(P)) = m + M + 1$$

for each vertex P in $\overline{\Gamma}(p,q)$ Thus, $\overline{\phi}$ induces a map on the subgraphs of $\overline{\Gamma}(p,q)$.

For each $i = 0, \ldots, N$, define

$$\breve{A}_i := \rho(\overline{\phi}(\Gamma_{N-i})),$$

and when $i > 0$, define

$$\check{V}_i := \rho(\overline{\phi}(\Upsilon_{N-i})).$$

(m1), (m2), (m3), (m4), and (m5) follow from proofs similar to the those used for (M1), (M2), (M3), (M4), and (M5). $\square$

5.3 An Inductive Proof of Lemma 5.2

5.3.1 Using Reductions for Induction

Suppose (p, q) is a co-prime pair with $q > 1$ and with $(p \bmod q) \neq 1$. By Lemma 4.6, $R(\overline{\Gamma})(p, q)$ is isomorphic to $\overline{\Gamma}(p^*, q^*)$ for some co-prime pair (p^*, q^*) so along with Lemma 5.9, $\overline{\Gamma}(p, q)$ can be simplified through a sequence of reductions and relative isomorphisms to $\overline{\Gamma}(p_0, q_0)$ such that $q_0 = 1$ or $(p \bmod q) = 1$.

Example 5.11.

$$\overline{\Gamma}(119, 43) \xrightarrow{R} \overline{\Gamma}(33, -23) \overset{rel.}{\cong} \overline{\Gamma}(33, 23) \xrightarrow{R} \overline{\Gamma}(10, 3)$$

The goal now is to show that when (p^*, q^*) is a pre-RTFN pair, (p, q) is also a pre-RTFN pair.

5.3.2 Leading and Trailing Vertices

Call a vertex in $\overline{\Gamma}(p, q)$ at the end of a $(\kappa + 1)$-segment a *leading vertex*, and any vertex at the beginning of a $(\kappa+1)$-segment a *trailing vertex* (see Figure 5.8). Let P be a leading vertex in $\overline{\Gamma}(p, q)$, and let Λ_L be the $(\kappa + 1)$-segment

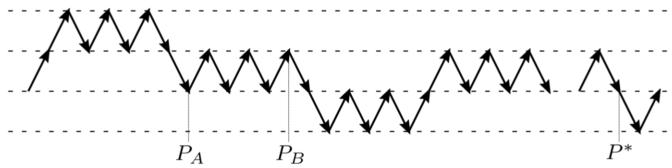


Figure 5.8: P_A is a leading vertex of $\overline{\Gamma}(13,11)$, and P_B is a trailing vertex of $\overline{\Gamma}(13,11)$ (left). $f_L(P_A) = f_T(P_B) = P^*$ in $R(\overline{\Gamma})(13,11)$ (right).

of $\overline{\Gamma}(p,q)$ immediately preceding P. Define $f_L(P)$ to be the vertex at the end of the edge in $R(\overline{\Gamma})(p,q)$ corresponding to Λ_L. Let P be a trailing vertex in $\overline{\Gamma}(p,q)$, and let Λ_T be the $(\kappa+1)$-segment of $\overline{\Gamma}(p,q)$ immediately following P. Define $f_T(P)$ to be the vertex at the beginning of the edge in $R(\overline{\Gamma})(p,q)$ corresponding to Λ_T.

f_L is a bijection from the leading vertices of $\Gamma(p,q)$ to the vertex set of $R(\overline{\Gamma})(p,q)$, and f_T is a bijection from the trailing vertices of $\Gamma(p,q)$ to the vertex set of $R(\overline{\Gamma})(p,q)$. Let P^* be a vertex in $R(\overline{\Gamma})(p,q)$. Since $f_L^{-1}(P^*)$ and $f_T^{-1}(P^*)$ are separated by a κ-block of length κ' or $\kappa'-1$, the gradings of $f_L^{-1}(P^*)$ and $f_T^{-1}(P^*)$ are either the same of differ by $\pm\kappa$.

Any vertex in $\overline{\Gamma}(p,q)$ at the end of a positive (or negative) segment is called a *peak* (resp. *valley*). There is a relationship between the gradings of the vertices in $\overline{\Gamma}(p,q)$ and $R(\overline{\Gamma})(p,q)$.

Proposition 5.12. *Let P and Q be leading vertices of $\overline{\Gamma}(p,q)$.*

1. *If P and Q are both peaks or both valleys, then*

$$\mathsf{gr}(f_L(P)) - \mathsf{gr}(f_L(Q)) = \mathsf{gr}(P) - \mathsf{gr}(Q).$$

2. If P is a valley and Q is a peak, then

$$\mathsf{gr}(f_L(P)) - \mathsf{gr}(f_L(Q)) = \mathsf{gr}(P) - \mathsf{gr}(Q) + \kappa.$$

3. If P is a peak and Q is a valley, then

$$\mathsf{gr}(f_L(P)) - \mathsf{gr}(f_L(Q)) = \mathsf{gr}(P) - \mathsf{gr}(Q) - \kappa.$$

Proof. This follows immediately from Lemma 4.9 by consider the unique path subgraph of $R(\overline{\Gamma})(p, q)$ beginning with $f_L(P)$ and ending $f_L(Q)$. $\square$

Corollary 5.13. *P is a leading summit of $\overline{\Gamma}(p, q)$ if and only if $f_L(P)$ is a summit of $R(\overline{\Gamma})(p, q)$.*

5.3.3 Proof of Lemma 5.2

We now have everything we need to show that every co-prime pair (p, q) with p positive and q odd is a pre-RTFN pair. For each co-prime pair, we need to find a positive integer N, subgraphs

$$\Gamma_0, \ldots, \Gamma_N$$

and

$$\Upsilon_1, \ldots, \Upsilon_N$$

and integers

$$n_1, \ldots, n_N$$

satisfying (R1),(R2),(R3),(R4), and (R5). We prove this using a strong induction starting with the base cases below.

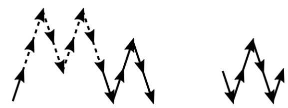


Figure 5.9: A graph $\overline{\Gamma}$ (left) with subgraph Υ (dashed) and $\overline{\Gamma} - \Upsilon$ (right).

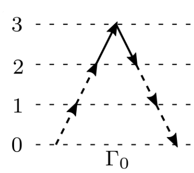
Given a subgraph Υ of a incremental cycle $\overline{\Gamma}$, define $\overline{\Gamma} - \Upsilon$ to be the incremental path obtained by removing the edges and the interior vertices of Υ from $\overline{\Gamma}$; see Figure 5.9.

Lemma 5.14. *Let (p, q) be a co-prime pair with p and q positive and q odd. If $q = 1$ or $(p \bmod q) = 1$ then (p, q) is a pre-RTFN pair.*

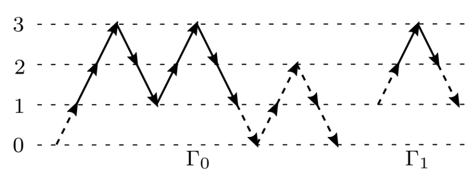
Proof. Define κ as in Proposition 4.3.

When $q = 1$, $\overline{\Gamma}(p, q)$ is the closure of a positive p-segment followed by a negative p-segment so $\overline{\Gamma}(p, q)$ only has one summit; see Figure 5.10a. It can be clearly seen that (p, q) is a pre-RTFN pair by making the following choice.

- Let $N = 1$.
- Let $\Gamma_0 = \Gamma(p, q)$.



(a) $\Gamma(3, 1)$ (left) has one summit. The solid arrows indicate Γ_1.



(b) $\Gamma(7, 3)$ (right) has two summits both in one 2-block of length 2. The solid arrows indicate Γ_1 and Γ_2 (in Γ_1).

Figure 5.10

- Let $\Gamma_1 = \Gamma_{\mathsf{top}}$.
- Let $n_1 = 1$.
- Let $\Upsilon_1 = \overline{\Gamma}(p, q) - \Gamma_{\mathsf{top}}$.

When $p \,\mathsf{mod}\, q = 1$, $\overline{\Gamma}(p, q)$ is the closure of a positive $(\kappa + 1)$-segment, a κ-block of length $q - 1$, a negative $(\kappa + 1)$-segment, followed by another κ-block of length $q - 1$ so $\overline{\Gamma}(p, q)$ has $(q + 1)/2$ summits all contained in the same κ-block; see Figure 5.10b.

Again, it's not hard to see that (p, q) is a pre-RTFN pair.

When $\kappa = 1$, make the following choices.

- Let $N = 1$.
- Let $\Gamma_0 = \Gamma(p, q)$.
- Let $\Gamma_1 = \Gamma_{\mathsf{top}}$.
- Let $n_1 = (q + 1)/2$.
- Let Υ_1 be the subgraph of $\overline{\Gamma}(p, q)$ with the all summits and their incident edges removed.

When $\kappa > 1$, make the following choices.

- Let $N = 2$.
- Let $\Gamma_0 = \Gamma(p, q)$.

- Let Γ_1 be a positive κ-segment followed by a negative κ-segment with a summit between them.
- Let $\Gamma_2 = \Gamma_{\mathsf{top}}$.
- Let Υ_1 be the subgraph of $\overline{\Gamma}(p,q)$ with the κ-block containing all the bottoms along with the edges immediately preceding and following the block.
- Let Υ_2 be $\mathsf{cl}(\Gamma_1) - \Gamma_{\mathsf{top}}$.
- Let $n_1 = (q+1)/2$.
- Let $n_2 = 1$.

$\square$

Let (p,q) be a co-prime pair with $q > 0$, and (p^*, q^*) be the co-prime pair defined by Lemma 4.6. Suppose (p^*, q^*) is a pre-RTFN pair so there is a positive integer N^* subgraphs

$$\Gamma_0^*, \ldots, \Gamma_N^*$$

and

$$\Upsilon_1^*, \ldots, \Upsilon_N^*$$

and integers

$$n_1^*, \ldots, n_N^*$$

satisfying (R1),(R2),(R3),(R4), and (R5).

Define κ and κ' as in (4.2) and (4.3) so $\overline{\Gamma}(p,q) \cong E(\overline{\Gamma}(p^*,q^*),\kappa,\kappa')$ by Proposition 4.11. For simplicity of notation, define

$$E(\Gamma^*) := E(\Gamma^*,\kappa,\kappa')$$

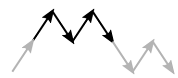
for any closable subgraph Γ^* of $\overline{\Gamma}(p^*,q^*)$.

To show that (p,q) is a pre-RTFN pair, we need to define N, the subgraphs $\{\Gamma_i\}_0^N$ and $\{\Upsilon_i\}_1^N$, and the integers $\{n_i\}_1^N$ for (p,q). This choice depends on how expansion effects the nested repeating pattern of summits in $\overline{\Gamma}(p^*,q^*)$.

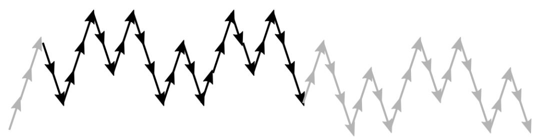
In general, we want to define Γ_i to be $E(\Gamma_i^*)$. By (R3), $(\Gamma_i^*)^{n_i^*}$ is a subgraph of Γ_{i-1}^* for all $i = 1,\ldots,N^*$. It follows that for all $i = 1,\ldots,N^*$, $E((\Gamma_i^*)^{n_i^*})$ is a subgraph of $E(\Gamma_{i-1}^*)$. We want $\Gamma_i^{n_i}$ to be a subgraph of Γ_{i-1} which is equal to $E(\Gamma_{i-1}^*)$. However, if Γ_i is $E(\Gamma_i^*)$, then $\Gamma_i^{n_i}$ is $(E(\Gamma_i^*))^{n_i^*}$, and $E((\Gamma_i^*)^{n_i^*})$ may not be equal to $(E(\Gamma_i^*))^{n_i^*}$. Nevertheless, $(E(\Gamma_i^*))^{n_i^*}$ is a subgraph of $E(\Gamma_{i-0}^*)$ by adding or removing κ edges.

While the leading summits of $\overline{\Gamma}(p,q)$ corresponds to the summits of $\overline{\Gamma}(p^*,q^*)$, we must also consider the non-leading summits in $\overline{\Gamma}(p,q)$. Let d be κ' or $\kappa'-1$ whichever is even. Let Γ_{top}^* be the subgraph of a summit in $\overline{\Gamma}(p^*,q^*)$ with its two adjacent vertices. $E(\Gamma_{\mathsf{top}}^*)$ is always the concatenation of a κ-block of even length, positive $(\kappa+1)$-segment, another κ-block of even length, and a negative $(\kappa+1)$-segment. It follows that every summit in $\overline{\Gamma}(p^*,q^*)$ corresponds to $d/2+1$ summits in $\Gamma(p,q)$.

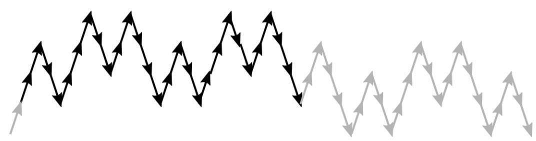
We define N, $\{\Gamma_i\}_0^N$, and $\{n_i\}_1^N$ as follows.



(a) The graph $R(\overline{\Gamma})(26,11) = \overline{\Gamma}(4,3)$ with $(\Gamma_1^*)^2$ in black and Υ_1^* in gray.



(b) The graph $\overline{\Gamma}(26,11) = E(\overline{\Gamma}(4,3))$ with $E((\Gamma_1^*)^2)$ in black and $E(\Upsilon_1^*)$ in gray.



(c) The graph $\Gamma_0 = \overline{\Gamma}(26,11)$ with $\Gamma_1^2 = (E(\Gamma_1^*))^2$ in black and Υ_1 in gray.

Figure 5.11: Expanding $\overline{\Gamma}(4,3)$ to $\overline{\Gamma}(26,11)$

Suppose $\kappa' = 1$ or $\kappa = 1$.

- Let $N = N^* + 1$.
- For each $i = 0, \ldots, N^*$, let $\Gamma_i = E(\Gamma_i^*)$.
- For each $i = 1, \ldots, N^*$, let $n_i = n_i^*$.
- Let $\Gamma_N = \Gamma_{\mathsf{top}}$.
- Let $n_N = d/2 + 1$.

Suppose $\kappa' > 1$ and $\kappa > 1$.

- Let $N = N^* + 2$.

- For each $i = 0, \ldots, N^*$, let $\Gamma_i = E(\Gamma_i^*)$.

- For each $i = 1, \ldots, N^*$, let $n_i = n_i^*$.

- Let Γ_{N-1} be a positive κ-segment followed by a negative κ-segment.

- Let $n_{N-1} = d/2 + 1$.

- Let $\Gamma_N = \Gamma_{\mathsf{top}}$.

- Let $n_N = 1$.

In either case, define $\Upsilon_i = \mathsf{cl}(\Gamma_{i-1}) - (\Gamma_i^{n_i})$ for $i = 1, \ldots, N$; see Figure 5.11.

Lemma 5.15. *The integers $\{n_i\}_1^N$ and the subgraphs $\{\Gamma_i\}_0^N$ and $\{\Upsilon_i\}_1^N$ satisfy (R1), (R2), (R3) and (R4).*

Proof. Since $\Gamma_0^* \cong \Gamma(p^*, q^*)$,

$$\Gamma_0 \cong E(\Gamma(p^*, q^*)) \cong \Gamma(p, q)$$

so (R1) is satisfied.

By definition, $\Gamma_N \cong \Gamma_{\mathsf{top}}$ so (R2) is satisfied.

For each $i = 1, \ldots, N$, $\Upsilon_i = \mathsf{cl}(\Gamma_{i-1}) - (\Gamma_i^{n_i})$ so

$$\mathsf{cl}(\Gamma_{i-1}) \cong \mathsf{cl}(\Gamma_i^{n_i} * \Upsilon_i).$$

Therefore, (R3) is satisfied.

When $i > N^*$, all of the summits in Γ_{i-1} are contained in $\Gamma_i^{n_i}$ by construction so $\Gamma_i = \Gamma_{i-1} - \Gamma_i^{n_i}$ has no summits.

For each $i = 1, \ldots, N^*$,

$$\Upsilon_i = \mathsf{cl}(\Gamma_{i-1}) - (\Gamma_i^{n_i}) = \mathsf{cl}(\Gamma_{i-1}) - (E(\Gamma_i^*)^{n_i^*})$$

and

$$E(\Upsilon_i^*) = \mathsf{cl}(E(\Gamma_{i-1}^*)) - E((\Gamma_i^*)^{n_i^*}) = \mathsf{cl}(\Gamma_{i-1}) - E((\Gamma_i^*)^{n_i^*}).$$

$E((\Gamma_i^*)^{n_i^*})$ is $(E(\Gamma_i^*))^{n_i^*}$ possibly with κ edges added of removed. It follows that Υ_i is $E(\Upsilon_i^*)$ with possibly κ edges added or removed; see Figure 5.11. Since no summits are in Υ_i^*, there are no summits $E(\Upsilon_i^*)$. The edges added or removed from $E(\Upsilon_i^*)$ to get Υ_i are not summits. Thus, there are no summits in Υ_i Therefore, (R4) is satisfied. □

Lemma 5.16. *The subgraphs $\{\Gamma_i\}_0^N$ satisfy (R5).*

Proof. First, we show what Γ_i has no bottoms for each $i = 1, \ldots, N$. Since $N^* \geqslant 1$, $\Gamma_1 = E(\Gamma_1^*)$. Since Γ_1^* has no bottoms, Γ_1 does not have bottoms. When $1 \leqslant i \leqslant N$,

$$\Gamma_{i-1} \cong \mathsf{cl}(\Gamma_i^{n_i} * \Upsilon_i)$$

so Γ_i is a subgraph of Γ_1 Therefore, Γ_i has no bottoms.

Suppose $0 \leqslant i \leqslant N$. Here we show that Γ_i is symmetric. When $i > N^*$, Γ_i is either the concatenation of a positive κ-segments and a negative κ-segment or Γ_{top}. In both case, Γ_i is clearly symmetric.

Suppose $0 \leqslant i \leqslant N^*$. In this case, $\Gamma_i = E(\Gamma_i^*)$. Our goal is to show that since Γ_i^* is symmetric, Γ_i is also symmetric.

Since Γ_i^* is symmetric, there is an order reversing bijection ϕ^* on the set of vertices of $\mathsf{cl}(\Gamma_i^*)$ and an integer k^* such that for each P^* in $\mathsf{cl}(\Gamma_i^*)$,

$$\mathsf{gr}(P^*) + \mathsf{gr}(\phi^*(P^*)) = k^*.$$

Let V_L and V_T be the sets of leading and trailing vertices of $\mathsf{cl}(\Gamma_i)$ respectively, and let V^* be the vertex set of $\mathsf{cl}(\Gamma_i^*)$. Define ϕ to be the unique order reversing bijection on the vertices of $\mathsf{cl}(\Gamma_i)$ such that the following diagram commutes,

$$\begin{tikzcd} V_L \arrow[r, "\phi|_{V_L}"] \arrow[d, "f_L"'] & V_T \arrow[d, "f_T"] \\ V^* \arrow[r, "\phi^*"] & V^* \end{tikzcd}$$

In particular, ϕ maps leading vertices bijectively to trailing vertices (see Figure 5.12). Let P_S be a leading summit of Γ_i, and let $P_S^* = f_L(P_S)$ in Γ_i^*.

Let $k = \mathsf{gr}(P_S) + \mathsf{gr}(\phi(P_S))$, and let P be an arbitrary vertex in Γ_i. The goal is to show that $\mathsf{gr}(P) + \mathsf{gr}(\phi(P)) = k$ which is done in four cases.

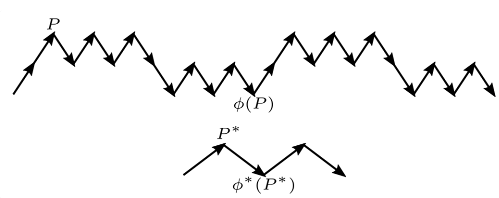


Figure 5.12: The incremental cycles $\mathsf{cl}(\Gamma_i)$ (top) and $\mathsf{cl}(\Gamma_i^*)$ (bottom) are shown. P is a leading vertex, and $f_L(P)$ is denoted P^*. $\phi(P)$ is a trailing vertex, and $\phi^*(P^*) = f_T(\phi(P))$.

Case 1. Suppose P is a leading vertex and $P^* := f_L(P)$ has the same type as P, either a peak (type $(-+)$) or valley (type $(+-)$). If P^* is of type $(-+)$, then $\phi^*(P^*)$ is of type $(+-)$, and if P^* is of type $(+-)$, then $\phi^*(P^*)$ is of type $(-+)$. Therefore, either $f_L^{-1}(P^*)$ and $f_T^{-1}(P^*)$ are both peaks and $f_L^{-1}(\phi^*(P^*))$ and $f_T^{-1}(\phi^*(P^*))$ are both valleys or $f_L^{-1}(P^*)$ and $f_T^{-1}(P^*)$ are both valleys and $f_L^{-1}(\phi^*(P^*))$ and $f_T^{-1}(\phi^*(P^*))$ are both peaks. In either case,

$$\mathsf{gr}(f_L^{-1}(\phi^*(P^*))) = \mathsf{gr}(f_T^{-1}(\phi^*(P^*))). \tag{5.17}$$

Thus,

$$\begin{aligned}
\mathsf{gr}(P) + \mathsf{gr}(\phi(P)) - k =& \mathsf{gr}(P) - \mathsf{gr}(P_S) + \mathsf{gr}(\phi(P)) - \mathsf{gr}(\phi(P_S)) \\
=& \mathsf{gr}(f_L^{-1}(P^*)) - \mathsf{gr}(f_L^{-1}(P_S^*)) \\
& + \mathsf{gr}(\phi(f_L^{-1}(P^*))) - \mathsf{gr}(\phi(f_L^{-1}(P_S^*))) \\
=& \mathsf{gr}(f_L^{-1}(P^*)) - \mathsf{gr}(f_L^{-1}(P_S^*)) \\
& + \mathsf{gr}(f_T^{-1}(\phi^*(P^*))) - \mathsf{gr}(f_T^{-1}(\phi^*(P_S^*)))
\end{aligned}$$

Summits are of type $(-+)$ so by (5.17),

$$\mathsf{gr}(f_T^{-1}(\phi^*(P^*))) - \mathsf{gr}(f_T^{-1}(\phi^*(P_S^*))) = \mathsf{gr}(f_L^{-1}(\phi^*(P^*))) - \mathsf{gr}(f_L^{-1}(\phi^*(P_S^*)))$$

By Proposition 5.12,

$$\begin{aligned}
\mathsf{gr}(P) + \mathsf{gr}(\phi(P)) - k =& \mathsf{gr}(f_L^{-1}(P^*)) - \mathsf{gr}(f_L^{-1}(P_S^*)) \\
& + \mathsf{gr}(f_L^{-1}(\phi^*(P^*))) - \mathsf{gr}(f_L^{-1}(\phi^*(P_S^*))) \\
=& \mathsf{gr}(P^*) - \mathsf{gr}(P_S^*) + \mathsf{gr}(\phi^*(P^*)) - \mathsf{gr}(\phi^*(P_S^*)) \\
=& \mathsf{gr}(P^*) + \mathsf{gr}(\phi^*(P^*)) - (\mathsf{gr}(P_S^*) + \mathsf{gr}(\phi^*(P_S^*))) \\
=& k^* - k^* = 0.
\end{aligned}$$

Therefore,

$$\mathsf{gr}(P) + \mathsf{gr}(\phi(P)) = k.$$

Case 2. Suppose P is a leading peak and $P^* := f_L(P)$ has type $(++)$. In this case, $f_L^{-1}(P^*)$ and $f_L^{-1}(\phi^*(P^*))$ are both peaks and $f_T^{-1}(P^*)$ and $f_T^{-1}(\phi^*(P^*))$ are both valleys. Thus,

$$\mathsf{gr}(f_L^{-1}(\phi^*(P^*))) = \mathsf{gr}(f_T^{-1}(\phi^*(P^*))) + \kappa,$$

and

$$\begin{aligned}
\mathsf{gr}(P) + \mathsf{gr}(\phi(P)) - k =& \mathsf{gr}(P) - \mathsf{gr}(P_S) + \mathsf{gr}(\phi(P)) - \mathsf{gr}(\phi(P_S)) \\
=& \mathsf{gr}(f_L^{-1}(P^*)) - \mathsf{gr}(f_L^{-1}(P_S^*)) \\
& + \mathsf{gr}(\phi(f_L^{-1}(P^*))) - \mathsf{gr}(\phi(f_L^{-1}(P_S^*))) \\
=& \mathsf{gr}(f_L^{-1}(P^*)) - \mathsf{gr}(f_L^{-1}(P_S^*)) \\
& + \mathsf{gr}(f_T^{-1}(\phi^*(P^*))) - \mathsf{gr}(f_T^{-1}(\phi^*(P_S^*))) \\
=& \mathsf{gr}(f_L^{-1}(P^*)) - \mathsf{gr}(f_L^{-1}(P_S^*)) \\
& + \mathsf{gr}(f_L^{-1}(\phi^*(P^*))) - \mathsf{gr}(f_L^{-1}(\phi^*(P_S^*))) - \kappa \\
=& \mathsf{gr}(P^*) - \mathsf{gr}(P_S^*) + \mathsf{gr}(\phi^*(P^*)) - \mathsf{gr}(\phi^*(P_S^*)) - \kappa + \kappa \\
=& 0.
\end{aligned}$$

Case 3. Suppose P is a leading valley and $P^* := f_L(P)$ has type $(--)$. In this case, $f_T^{-1}(P^*)$ and $f_T^{-1}(\phi^*(P^*))$ are both peaks and $f_L^{-1}(P^*)$ and $f_L^{-1}(\phi^*(P^*))$ are both valleys. Thus,

$$\mathsf{gr}(f_L^{-1}(\phi^*(P^*))) = \mathsf{gr}(f_T^{-1}(\phi^*(P^*))) - \kappa,$$

and

$$
\begin{aligned}
\mathsf{gr}(P) + \mathsf{gr}(\phi(P)) - k =& \mathsf{gr}(P) - \mathsf{gr}(P_S) + \mathsf{gr}(\phi(P)) - \mathsf{gr}(\phi(P_S)) \\
=& \mathsf{gr}(f_L^{-1}(P^*)) - \mathsf{gr}(f_L^{-1}(P_S^*)) \\
& + \mathsf{gr}(\phi(f_L^{-1}(P^*))) - \mathsf{gr}(\phi(f_L^{-1}(P_S^*))) \\
=& \mathsf{gr}(f_L^{-1}(P^*)) - \mathsf{gr}(f_L^{-1}(P_S^*)) \\
& + \mathsf{gr}(f_T^{-1}(\phi^*(P^*))) - \mathsf{gr}(f_T^{-1}(\phi^*(P_S^*))) \\
=& \mathsf{gr}(f_L^{-1}(P^*)) - \mathsf{gr}(f_L^{-1}(P_S^*)) \\
& + \mathsf{gr}(f_L^{-1}(\phi^*(P^*))) - \mathsf{gr}(f_L^{-1}(\phi^*(P_S^*))) + \kappa \\
=& \mathsf{gr}(P^*) - \mathsf{gr}(P_S^*) + \mathsf{gr}(\phi^*(P^*)) - \mathsf{gr}(\phi^*(P_S^*)) + \kappa - \kappa \\
=& 0.
\end{aligned}
$$

Case 4. Suppose P is not a leading vertex. Let P' be the leading vertex in $\mathsf{cl}(\Gamma_i)$ such that the length of the path $\omega(P', P)$, the path in $\mathsf{cl}(\Gamma_i)$ from P' to P, is minimal. It follows that $\omega(P', P)$ is isomorphic to a subgraph of a κ-block as in Figure 5.13. In particular, there are no leading vertices between P' and P in $\mathsf{cl}(\Gamma_i)$. Therefore, there are no trailing vertices between $\phi(P)$ and $\phi(P')$ in $\mathsf{cl}(\Gamma_i)$ so $\omega(\phi(P), \phi(P'))$, the path from $\phi(P)$ to $\phi(P')$ in $\mathsf{cl}(\Gamma_i)$, is also isomorphic to a subgraph of a κ-block.

Let Q be the closest vertex to P with grading $\mathsf{gr}(Q) = \mathsf{gr}(P')$. When P' is a peak, Q is a peak. Likewise, when P' is a valley, Q is a valley. Define δ be the distance from P' to Q. Since P is in a κ-block which starts at P‘, Q

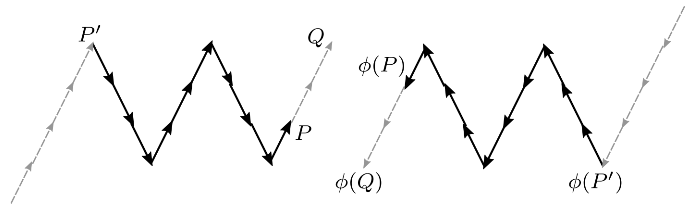


Figure 5.13: $\omega(P', P)$ (left) and $\omega(\phi(P), \phi(P'))$ (right) are shown in solid black. The dashed gray arrows are other edges in $\mathsf{cl}(\Gamma_i$. The case shown is when P' is a peak.

and P lie on the same segment so

$$\mathsf{gr}(Q) - \mathsf{gr}(P) = \begin{cases} \delta & \text{when } Q \text{ is a peak} \\ -\delta & \text{when } Q \text{ is a valley} \end{cases}.$$

also, $\phi(Q)$ and $\phi(P)$ lie on the same segment so

$$\mathsf{gr}(\phi(Q)) - \mathsf{gr}(\phi(P)) = \begin{cases} -\delta & \text{when } Q \text{ is a peak} \\ \delta & \text{when } Q \text{ is a valley} \end{cases}.$$

If P' and Q are peaks, then

$$\mathsf{gr}(P) = \mathsf{gr}(Q) - \delta = \mathsf{gr}(P') - \delta$$

and

$$\mathsf{gr}(\phi(P)) = \mathsf{gr}(\phi(Q)) + \delta = \mathsf{gr}(\phi(P')) + \delta.$$

If P' and Q are valleys, then

$$\mathsf{gr}(P) = \mathsf{gr}(Q) + \delta = \mathsf{gr}(P') + \delta$$

and

$$\mathsf{gr}(\phi(P)) = \mathsf{gr}(\phi(Q)) - \delta = \mathsf{gr}(\phi(P')) - \delta.$$

In both cases,

$$\mathsf{gr}(P) + \mathsf{gr}(\phi(P)) = \mathsf{gr}(P') + \mathsf{gr}(\phi(P')) = k.$$

Therefore, for every vertex P in $\mathsf{cl}(\Gamma_i)$, $\mathsf{gr}(P) + \mathsf{gr}(\phi(P)) = k$ so Γ_i is symmetric.

$\square$

Proof of Lemma 5.2. By Lemma 5.10, it is sufficient to show that every co-prime pair is a pre-RTFN pair.

Let (p, q) be a co-prime pair with p positive and q odd. If $q = 1$ or $(p \,\mathsf{mod}\, q) = 1$ with q positive, then (p, q) is a pre-RTFN pair by Lemma 5.14. If $q = -1$ then (p, q) is a pre-RTFN pair by Lemma 5.9.

Suppose $|q| \neq 1$ and $(p \,\mathsf{mod}\, q) > 1$, and assume every co-prime pair (p', q') with $|q'| < |q|$ is a pre-RTFN pair. When q is positive, define the co-prime pair (p^*, q^*) as in Lemma 4.6. Since $|q^*| < |q|$, (p^*, q^*) is a pre-RTFN pair. By Lemma 5.15 and Lemma 5.16, (p, q) is also pre-RTFN pair. When q is negative, the pair $(p, -q)$ is a pre-RTFN pair by the above argument. Thus (p, q) is a pre-RTFN pair by Lemma 5.9.

By strong induction, every co-prime pair (p, q) with p positive and q odd is a pre-RTFN pair. $\square$

Chapter 6

Residual Torsion-Free Nilpotence, Bi-Orderability and Pretzel Knots

6.1 Mayland's Technique

Mayland used a description of the commutator subgroup of a knot group to investigate when the group is residually finite [35]. In this section, we show how Mayland's technique can be used to find a sufficient condition for the commutator subgroup of a knot group to be residually torsion-free nilpotent. This is the technique Mayland and Murasugi [36] used to show pseudo alternating links whose Alexander polynomials have relatively prime coefficients satisfied Baumslag's conditions.

6.1.1 The Free Factor Property and the Commutator Subgroup

Let J be a knot in S^3, and suppose J bounds a minimal genus Seifert surface S such that S is *unknotted*, in other words, $\pi_1(S^3 \backslash S)$ is a free group. Let $\hat{S} = M_J \cap S$. Let G be the commutator subgroup of $\pi_1(M_J)$.

Let U be the image of a bi-collar embedding $\hat{S} \times [-1, 1] \hookrightarrow M_J$ where $\hat{S}$ is the image of $\hat{S} \times \{0\}$, and let $M_S = M_J \backslash \hat{S}$. Denote images of $\hat{S} \times (0, 1]$ and $\hat{S} \times [-1, 0)$ in M_S as U^+ and U^- respectively. Let $X = \pi_1(M_S)$ which is a free group of rank $2g$ where g is the genus of J. Consider the inclusion maps

$i^+ : U^+ \to M_S$ and $i^- : U^- \to M_S$. Let H be the image of the induced map $i^+_* : \pi_1(U^+) \to \pi_1(M_S)$ and K be the image of $i^-_* : \pi_1(U^-) \to \pi_1(M_S)$.

For each integer n, let X_n be a copy of X, $H_n \subset X_n$ be a copy of H, and $K_n \subset X_n$ be a copy of K. The fundamental groups of U, U^+ and U^- are canonically isomorphic, and since S has minimal genus, i^+_* and i^-_* are injective. Therefore, H_n and K_{n+1} are identified with a rank $2g$ free group F. By a result of Brown and Crowell [6, Theorem 2.1], G is an amalgamated free product of the following form.

$$G \cong \cdots \underset{F}{*} X_{-2} \underset{F}{*} X_{-1} \underset{F}{*} X_0 \underset{F}{*} X_1 \underset{F}{*} X_2 \underset{F}{*} \cdots \tag{6.1}$$

For each non-negative integer m, define Z^m as follows.

$$Z^m := X_{-m} \underset{F}{*} X_{1-m} \underset{F}{*} \cdots \underset{F}{*} X_{m-1} \underset{F}{*} X_m \tag{6.2}$$

The direct limit of the Z^m's is isomorphic to G. Furthermore, since i^+_* and i^-_* are injective, the natural inclusion $Z^m \hookrightarrow Z^{m+1}$ is an embedding so G is an ascending chain of subgroups as follows.

$$Z^0 < Z^1 < Z^2 < \cdots < Z^m < \cdots < G = \bigcup_{m=1}^{\infty} Z^m$$

A subgroup A of a free group B is a *free factor* if $B = A * D$ for some subgroup D of B. It immediately follows that A is a free factor of B if and only if every (equivalently, at least one) free basis of A extends to a free basis of B. A theorem of Mayland [35, Theorem 3.2] provides the following sufficient conditions for each Z^m to be parafree.

Proposition 6.1. *[35, Theorem 3.2] If H and K are free factors of $H[X,X]$ and $K[X,X]$ respectively and $|X : H[X,X]| = |X : K[X,X]| = p^l$ for some prime p and non-negative integer l, then for every non-negative m, Z^m is parafree of rank $2g$.*

The knot J is *rationally homologically fibered* if the induced map on homology, $i_h^+ : H_1(U^+;\mathbb{Q}) \to H_1(M_S;\mathbb{Q})$ (or equivalently $i_h^- : H_1(U^-;\mathbb{Q}) \to H_1(M_S;\mathbb{Q})$), is an isomorphism. Let S_+ be a Seifert matrix representing i_h^+ so that $S_- := S_+^T$ is a Seifert matrix representing i_h^-. S_+ is also a presentation matrix for the abelian group $X/H[X,X]$. Similarly, S_- is a presentation matrix for $X/K[X,X]$. Thus,

$$\frac{X}{H[X,X]} \cong \frac{X}{K[X,X]}. \tag{6.3}$$

For some non-negative integer k,

$$t^k\Delta_J(t) = \det(tS_+ - S_+^T) = d_0 + d_1 t + \cdots + d_{2g}t^{2g}.$$

It is a well known fact that $d_i = d_{2g-i}$ (see [39, Chapter 6]).

Proposition 6.2. *Suppose J is a knot in S^3. The following statements are equivalent:*

(a) J is rationally homologically fibered.

(b) $|X : H[X,X]|$ is finite.

(c) $|X : K[X,X]|$ is finite.

(d) $\deg \Delta_J = 2g$.

Proof. The equivalence of (b) and (c) follows from (6.3).

Since S_+ is a presentation matrix for $X/H[X,X]$, $|X : H[X,X]|$ is finite if and only if $|\det(S_+)| \neq 0$. It follows that (a) and (b) are equivalent.

Since $d_{2g} = d_0 = \det(S_+)$, $\deg \Delta_J = 2g$ if and only if $\det(S_+) \neq 0$ so (a) and (d) are equivalent. $\square$

Proposition 6.3. *When J is rationally homologically fibered,*

$$|X : H[X,X]| = |X : K[X,X]| = |\Delta_J(0)|.$$

Proof. When J is rationally homologically fibered,

$$|X : H[X,X]| = |\det(S_+)| = |\Delta_J(0)|$$

so the proposition follows from (6.3). $\square$

For each non-negative m,

$$\frac{Z^{m+1}}{Z^m[Z^{m+1}, Z^{m+1}]} \cong \frac{X}{H[X,X]} \times \frac{X}{K[X,X]}$$

so when J is rationally hmologically fibered,

$$|Z^{m+1} : Z^m[Z^{m+1}, Z^{m+1}]| = |X : H[X,X]||K : H[X,X]| = \Delta_J(0)^2 \quad (6.4)$$

by Proposition 6.3.

Definition 6.4. The Seifert surface S is said to *satisfy the free factor property* if H and K are free factors of $H[X,X]$ and $K[X,X]$ respectively.

Note that this property is independent of the orientation of S. A sufficient condition for the residual torsion-free nilpotence of G can be summarized as follows.

Proposition 6.5. *Suppose J is a rationally homologically fibered knot in S^3 with unknotted minimum genus Seifert surface S. If S satisfies the free factor property and $|\Delta_J(0)|$ is a prime power, then the commutator subgroup, G, is residually torsion-free nilpotent.*

Proof. Suppose J is a rationally homologically fibered with unknotted minimum genus Seifert surface S satifying the free factor property, and suppose $|\Delta_J(0)|$ is a prime power.

Define Z^m for each non-negative integer m as in (6.2). By Proposition 6.3, $|X : H[X, X]|$ and $|K : H[X, X]|$ are prime powers since J is rationally homologically fibered. Thus, by Proposition 6.1, each Z^m is parafree of rank twice the genus of J.

By (6.4),

$$|Z^{m+1} : Z^m[Z^{m+1}, Z^{m+1}]| = \Delta_J(0)^2$$

so

$$|Z^{m+1} : Z^m[Z^{m+1}, Z^{m+1}]|$$

is finite. Therefore, by Proposition 2.7, G is residually torsion-free nilpotent. $\square$

6.1.2 Pseudo-Alternating Links

A *special alternating diagram* is an alternating link diagram in which all of the crossings have the same sign. Any link with such a diagram is called a *special alternating link*. The Seifert surface described by performing Seifert's algorithm on special alternating diagram is a *primitive flat surface*. A *generalized flat surface* is any surface which can be obtained by combining some number of primitive flat surfaces by Murasugi sums. A link which bounds a generalized flat surface is a *pseudo-alternating link*. Alternating links are pseudo-alternating. However, all torus links, many of which are not alternating, are also pseudo-alternating links.

Pseudo-alternating knots are rationally homologically fibered knots and bound surfaces satisfing the free factor condition [36, Theorem 2.5]. Therefore, the knot group of a pseudo-alternating knot, whose Alexander polynomial has a prime power leading coefficient, has residually torsion-free nilpotent commutator subgroup.

6.2 Genus One Pretzel Knots

Let J be the $P(2p+1, 2q+1, 2r+1)$ pretzel knot for some integers p, q, and r with $1 \leqslant q \leqslant r$ and $p \neq -1$ or 0. Let S be the unknotted genus 1 surface depicted in Figure 6.1, which we refer to as the *standard Seifert surface* of J. For the genus one pretzel knots which aren't two-bridge knots, the standard Seifert surface is the unique Seifert surface of minimal genus up to isotopy and [16].

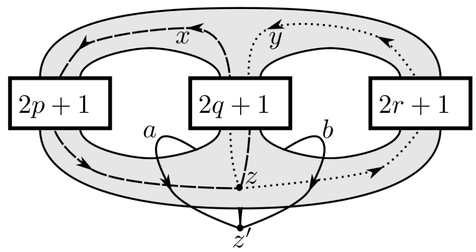


Figure 6.1: The Seifert surface S of $P(2p+1, 2q+1, 2r+1)$.

In this section, we show analyze when S satisfies the free factor property. When $p > 0$, $P(2p+1, 2q+1, 2r+1)$ is an alternating knot, and thus, $P(2p+1, 2q+1, 2r+1)$ is pseudo-alternating. However, this is not true when $p \leqslant -2$.

Proposition 6.6. *When* $1 \leqslant q \leqslant r$ *and* $p \leqslant -2$*, the pretzel knot* $P(2p+1, 2q+1, 2r+1)$ *is not a pseudo-alternating knot.*

Proof. Suppose $P(2p+1, 2q+1, 2r+1)$ is a pseudo-alternating knot. When $1 \leqslant q \leqslant r$ and $p \leqslant -2$, the diagram given in Figure 1.2 has a minimal number of crossings [29, Theorem 10]. Since this diagram is not alternating, $P(2p+1, 2q+1, 2r+1)$ cannot be alternating by a theorem of Kauffman, Murasugi, and Thistlethwaite [26, 27, 38, 51]. In particular, $P(2p+1, 2q+1, 2r+1)$ is not special alternating. Thus, $P(2p+1, 2q+1, 2r+1)$ must be the boundary of a surface S which is the Murasugi sum of two generalized flat surfaces, S_1 and S_2, which aren't disk.

By Gabai [15], S must be a minimal genus Seifert surface so $\chi(S) = -1$.

Analyzing the effect of a Murasugi sum on the Euler characteristic yields

$$-1 = \chi(S) = \chi(S_1) + \chi(S_2) - 1.$$

Since S_1 and S_2 are not disks, neither S_1 or S_2 has positive Euler characteristic. It follows that $\chi(S_1) = \chi(S_2) = 0$ so S_1 and S_2 are both annuli.

The boundary of a Murasugi sum of two annuli is a double twist knot which is alternating. Thus, $P(2p + 1, 2q + 1, 2r + 1)$ is alternating, which is a contradiction. $\square$

Define M_J, M_S, X, H and K as in section 6.1. Here we offer a concrete description of the maps on fundamental groups i_*^+ and i_*^- for genus one pretzel knots. This is the same discription used by Crowell and Trotter in [12]. Choose a base point z on the lower part of S, and let x and y be the classes generating $\pi_1(S, z)$ represented by the loops indicated in Figure 6.1. Let z^+ and z^- be pushoffs of z of each side of S. Let z' be the base point of M_S obtained by shifting z tangentially along S through ∂S. Let δ^+ and δ^- be arcs connecting z to z^+ and z^- respectively; see Figure 6.2. Finally, let a and b be the indicated classes generating $\pi_1(M_S, z')$.

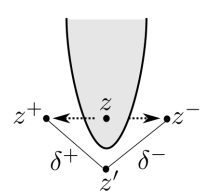


Figure 6.2: Isotopy of base points

By slightly isotoping elements of $\pi_1(S,z)$ off of S, $\pi_1(U^+,z^+)$ and $\pi_1(U^-,z^-)$ are canonically identified to $\pi_1(S,z)$ which is a rank two free group, F, generated by x and y. The group $X := \pi_1(M_S, z')$ is a rank two free group generated by a and b. The map $i^+_* : F \to X$ takes a class $[\gamma]$ in $\pi_1(U^+,z^+) = F$ to the class $[\delta^+ * \gamma * (-\delta^+)]$ in $\pi_1(M_S,z') = X$. Likewise, the map $i^-_* : F \to X$ takes $[\gamma]$ to $[\delta^- * \gamma * (-\delta^-)]$.

With these choices, we define the elements

$$\begin{aligned} \alpha_H &:= i^+_*(x) = (b^{-1}a)^{q+1}a^p & \qquad \alpha_K &:= i^-_*(x) = (ab^{-1})^q a^{p+1} \\ \beta_H &:= i^+_*(y) = b^{r+1}(a^{-1}b)^q & \qquad \beta_K &:= i^-_*(y) = b^r(ba^{-1})^{q+1} \end{aligned} \tag{6.5}$$

so that

$$H = \langle\{\alpha_H, \beta_H\}\rangle \quad K = \langle\{\alpha_K, \beta_K\}\rangle.$$

Thus, the Seifert matrices for i^+_* and i^-_* are

$$S_+ = \begin{pmatrix} p+q+1 & -q-1 \\ -q & q+r+1 \end{pmatrix} \text{ and } S_- = \begin{pmatrix} p+q+1 & -q \\ -q-1 & q+r+1 \end{pmatrix}. \tag{6.6}$$

Let $N = \det S_+ = \det S_-$. Up to multiplication by a signed power of t, the Alexander polynomial of J is

$$\Delta_J(t) = Nt^2 + (1-2N)t + N.$$

Δ_J has two positive real roots when $N < 0$ and two non-real roots when $N > 0$. When $N = 0$, $\Delta_J(t) = 1$. Therefore, we have the following proposition.

Proposition 6.7. *Let J be the $P(2p+1, 2q+1, 2r+1)$ pretzel knot, and define N as above. When the commutator subgroup of $\pi_1(M_J)$ is residually* *torsion-free nilpotent and $N < 0$, $\pi_1(M_J)$ is bi-orderable. When $N > 0$, $\pi_1(M_J)$ is never bi-orderable, regardless of whether or not the commutator subgroup of $\pi_1(M_J)$ is residually torsion-free nilpotent.*

When $N \neq 0$, J is rationally homologically fibered by Proposition 6.2. Simply considering the integer N can provide useful information about residual torsion-free nilpotence.

Proposition 6.8. *When $N = 0$, G is not residually torsion-free nilpotent.*

Proof. $\Delta_J(t) = 1$ when $N = 0$ so G cannot be residually nilpotent by Proposition 2.11. $\square$

Proposition 6.9. *If $|N| = 1$, then standard Seifert surface S does not satisfy the free factor property.*

Proof. Suppose S satisfies the free factor property. Define X, H, and K as in section 6.1. Each of these are rank 2 free groups.

When $|N| = 1$, $X/H[X, X] \cong X/K[X, X] \cong 1$ by Proposition 6.3 so $X = H[X, X] = K[X, X]$. Since H is a free factor of $H[X, X]$ and both are rank 2 free groups, $H = H[X, X] = X$. Similarly, since K is a free factor of $K[X, X]$ and both are rank 2 free groups, $K = X$. This implies that i_*^+ and i_*^- are isomorphisms. Thus, $\pi_1(M_J)$ is an extension $\mathbb{Z}$ described by the following short exact sequence.

$$1 \to X \to \pi_1(M_J) \to \mathbb{Z} \to 1$$

The Stallings fibration theorem [49] implies that J is a genus 1 fibered knot [49]. However, the only genus 1 fibered knots are the trefoil and the figure 8 knot [7, 17] which is a contradiction since we are assuming J is not a two-bridge knot. $\square$

In light of Proposition 6.5, to prove the commutator subgroup of $\pi_1(M_J)$ is residually torsion-free nilpotent, it is sufficient to show S satisfies the free factor property.

6.3 Verifying the Free Factor Property

6.3.1 Outline

We use a basic procedure, outlined below, to analyze whether or not S satisfies the free factor property.

1. Find a presentation matrix for $X/H[X,X]$ of the form
$$\begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \text{ or } \begin{pmatrix} u & 0 \\ v & w \end{pmatrix}$$
using row operations. Note, u and w can always be made positive. Thus, $X/H[X,X]$ is isomorphic to $(\mathbb{Z}/u\mathbb{Z}) \times (\mathbb{Z}/w\mathbb{Z})$. The $\mathbb{Z}/u\mathbb{Z}$ factor is generated by the class of a in $X/H[X,X]$, and the $\mathbb{Z}/w\mathbb{Z}$ factor is generated by the class of b.

2. Since $X/H[X,X]$ is abelian, the set $\mathcal{C}$, defined below, is a set of coset representatives of $H[X,X]$.
$$\mathcal{C} = \{a^k b^l : 0 \leqslant k < u, 0 \leqslant l < w\}$$

Given $x \in X$, denote by $\overline{x}$, the coset representative of x in $\mathcal{C}$. Define

$$x_{c,x} := cx(\overline{cx})^{-1}$$

where $c \in \mathcal{C}$ and $x \in \{a, b\}$. From this, we find the following free basis for $H[X, X]$ using the Reidemeister-Schreier method; see [25] for details.

$$\mathcal{B} = \{x_{c,x} : c \in \mathcal{C}, x \in \{a, b\}, x_{c,x} \neq 1\}$$

3. Use the Reidemeister-Schreier rewriting process to rewrite the generating set of H from (6.5). A word $\alpha \in H$, where $\alpha = \alpha_1^{s_1} \cdots \alpha_k^{s_k}$ with $\alpha_i \in \{a, b\}$ and $s_i = \pm 1$, can be rewritten as

$$\alpha = x_{c_1,\alpha_1}^{s_1} \cdots x_{c_k,\alpha_k}^{s_k}$$

where

$$c_i = \begin{cases} \overline{\alpha_1 \cdots \alpha_{i-1}} & \text{when } s_i = 1 \\ \overline{\alpha_1 \cdots \alpha_i} & \text{when } s_i = -1 \end{cases}.$$

4. Determine if the generating set of H can be extended to a free basis of $H[X, X]$.

5. Repeat this procedure for K.

When the free bases of H and K can be extended to free bases of $H[X, X]$ and $K[X, X]$ respectively, S satisfies the free factor property. If the chosen basis of either H or K fails to extend then S cannot satisfy the free factor property. Here's an example below.

Lemma 6.10. *If J is the $P(-5,7,9)$ pretzel knot then S satisfies the free factor property.*

Proof. Suppose J is $P(-5,7,9)$. From (6.5), we have that

$$\begin{array}{ll} \alpha_H = (b^{-1}a)^4a^{-3} & \alpha_K = (ab^{-1})^3a^{-2} \\ \beta_H = b^5(a^{-1}b)^3 & \beta_K = b^4(ba^{-1})^4 \end{array}.$$

The abelian group $X/H[X,X]$ has presentation matrix

$$\begin{pmatrix} 1 & -4 \\ -3 & 8 \end{pmatrix}$$

which becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

after row operations.

From this, we get $\mathcal{C} = \{1, b, b^2, b^3\}$ as a set of coset representatives of $H[X,X]$. We apply Reidemeister-Schreier to obtain the following free basis of $H[X,X]$.

$$\mathcal{B} = \{a, bab^{-1}, b^2ab^{-2}, b^3ab^{-3}, b^4\}$$

Label the basis elements as follows: $x_k := b^kab^{-k}$ for $0 \leqslant k \leqslant 3$ and $x_4 := b^4$.

Now, we can rewrite α_H and β_H in terms of $\mathcal{B}$.

$$\begin{aligned} \alpha_H =& (b^{-1}a)^4a^{-3} \\ =& (b^{-4})(b^3ab^{-3})(b^2ab^{-2})(bab^{-1})(a^{-1})(a^{-1}) \\ =& x_4^{-1}x_3x_2x_1x_0^{-2} \end{aligned}$$

and

$$
\begin{aligned}
\beta_H =& b^4(a^{-1}b)^3 \\
=& (b^{-4})(ba^{-1}b^{-1})(b^2a^{-1}b^{-2})(b^3a^{-1}b^{-3})(b^4) \\
=& x_4x_1^{-1}x_2^{-1}x_3^{-1}x_4.
\end{aligned}
$$

Thus,

$$x_4 = \beta_H\alpha_H x_0^2$$

and

$$x_3 = x_4\alpha_H x_0^2 x_1^{-1} x_2^{-1}.$$

Therefore, the set

$$\{\alpha_H, \beta_H, x_0, x_1, x_2\}$$

is a generating set of five elements for $H[X, X]$, and thus, is a free basis. It follows that

$$H[X, X] = H * \{x_0, x_1, x_2\}$$

so H is a free factor of $H[X, X]$.

After row reductions, $X/K[X, X]$ has presentation matrix

$$\begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix}.$$

From this we get a free basis of $K[X, X]$ as follows.

$$x_k := \begin{cases} ab^{-3} & \text{for } k = 0 \\ b^k ab^{1-k} & \text{for } 1 \leqslant k \leqslant 3 \\ b^4 & \text{for } k = 4 \end{cases}$$

Rewriting α_K and β_K, we get

$$\alpha_K = (ab^{-3})(b^2ab^{-1})(ab^{-3})(b^2a^{-1}b^{-3})(b^3a^{-1}) = x_0x_2x_0x_3^{-1}x_0^{-1}$$

and

$$\beta_K = (b^4)(ba^{-1}b^{-2})(b^3a^{-1})(ba^{-1}b^{-2})(b^3a^{-1}) = x_4x_2^{-1}x_0^{-1}x_2^{-1}x_0^{-1}.$$

Thus,

$$x_3 = x_0^{-1}\alpha_K^{-1}x_0x_2x_0$$

and

$$x_4 = \beta_K(x_0x_2)^2.$$

Therefore, the set

$$\{\alpha_K, \beta_K, x_0, x_1, x_2\}$$

is a free basis of $K[X, X]$ so K is a free factor of $K[X, X]$. Therefore, S satisfies the free factor property. □

6.4 The Free Factor Property, Bi-Orderability and the $P(-3, 2q+1, 2r+1)$ Pretzel Knots

Lemma 6.11. *If J is a $P(-3, 3, 2r+1)$ pretzel knot then S satisfies the free factor property.*

Proof. From (6.5), we have that

$$\begin{array}{ll} \alpha_H = b^{-1}ab^{-1}a^{-1} & \alpha_K = ab^{-1}a^{-1} \\ \beta_H = b^{r+1}a^{-1}b & \beta_K = b^{r+1}a^{-1}ba^{-1} \end{array}.$$

The abelian group $X/H[X,X]$ has presentation matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

when r is even and

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

when r is odd.

Using $\mathcal{C} = \{1, b\}$ as a set of coset representatives, we obtain a free basis of $H[X,X]$, $\mathcal{B} = \{x_0, x_1, x_2\}$, by applying Reidemeister-Schreier.

When r is even,

$$x_0 = a, x_1 = bab^{-1}, \text{ and } x_2 = b^2$$

so

$$\alpha_H = (b^{-2})(bab^{-1})(a^{-1}) = x_2^{-1}x_1x_0^{-1}$$

and

$$\beta_H = (b^{2k})(ba^{-1}b^{-1})(b^2) = x_2^k x_1^{-1} x_2$$

where $r = 2k$.

When r is odd,

$$x_0 = ab^{-1}, x_1 = ba, \text{ and } x_2 = b^2$$

so

$$\alpha_H = (b^{-2})(ba)(b^{-2})(ba^{-1}) = x_2^{-1}x_1x_2^{-1}x_0^{-1}$$

and

$$\beta_H = (b^{2k+2})(a^{-1}b^{-1})(b^2) = x_2^{k+1}x_1^{-1}x_2$$

where $r = 2k+1$.

In either case, the set $\{\alpha_H, \beta_H, x_2\}$ is a free basis of $H[X,X]$ so H is a free factor of $H[X,X]$.

$X/K[X,X]$ has presentation matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using $\mathcal{C} = \{1, a\}$ as a set of coset representatives, we get the free basis of $K[X,X]$, $\mathcal{B} = \{x_0, x_1, x_2\}$ where

$$x_0 = a^2, x_1 = b \text{ and } x_2 = aba^{-1}.$$

Thus,

$$\alpha_K = x_2^{-1} \text{ and } \beta_K = x_1^{r+1}x_0^{-1}x_2.$$

The set $\{\alpha_K, \beta_K, x_1\}$ is a free basis of $K[X,X]$ so K is a free factor of $K[X,X]$. Therefore, S satisfies the free factor property. □

Lemma 6.12. *If J is $P(1-2q, 2q+1, 2q^2+1)$ or $P(1-2q, 2q+1, 2q^2-3)$ then S, the standard Seifert surface of J, satisfies the free factor property.*

Proof. When J is $P(1-2q, 2q+1, 2q^2+1)$, $p = -q$ and $r = q^2$. When J is $P(1-2q, 2q+1, 2q^2-3)$, $p = -q$ and $r = q^2-2$. In both cases, $|N| = 1$ so by Proposition 6.9, S does not satisfy the free factor property. □

The following results obtained in one of our papers [23], but the proofs were omitted here.

Lemma 6.13. *Suppose J is $P(-3, 2q+1, 2r+1)$ and one of the following conditions hold:*

1. *$q = 2$ and $r \geqslant 6$,*

2. *$q = 3$ and $r \geqslant 4$,*

3. *$q > 3$.*

Then, S satisfies the free factor property.

Lemma 6.14. *If J is $P(-3, 5, 7)$ then $\pi_1(J)$ does not have residually torsion-free nilpotent commutator subgroup.*

Proof. For $P(-3, 5, 7)$, $N = 0$ so this follows from Proposition 6.8. □

Lemma 6.15. *Suppose J is $P(-3, 5, 11)$ or $P(-3, 7, 7)$. Then S the standard Seifert surface of J does not satisfy the free factor property.*

Proof. When J is $P(-3, 5, 11)$, $X/H[X, X]$ has presentation matrix

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}.$$

We have the free basis $\mathcal{B} = \{ab^{-1}, bab^{-2}, b^2\}$ of $H[X, X]$. Let $x_0 = ab^{-1}$, $x_1 = bab^{-2}$, and $x_2 = b^2$ so

$$\beta_H = b^6(a^{-1}b)^2 = x_2^2 x_1^{-2} x_2.$$

Let

$$\Gamma := \frac{H[X,X]}{\langle \beta_H^{H[X,X]} \rangle} \cong \langle x_0, x_1, x_2 : x_2^3 x_1^{-2} \rangle$$

where $\langle \beta_H^{H[X,X]} \rangle$ is the normal closure of β_H in $H[X,X]$. Suppose $\{\alpha_H, \beta_H\}$ could be extended to a basis of $H[X,X]$, then Γ is a free group. Γ has as a subgroup isomorphic to $E := \langle x_1, x_2 : x_2^3 x_1^{-2} \rangle$. The abelianization of E is $\mathbb{Z}$, but E is not abelian since x_1 and x_2 do not commute. Thus, E is not free, and Γ isn't free either, which is a contradiction.

Therefore, H is not a free factor of $H[X,X]$, and S does not satisfy the free factor property.

When J is $P(-3,7,7)$,

$$\beta_H = b^4 (a^{-1}b)^3 = x_2 x_1^{-3} x_2$$

where $x_0 = ab^{-1}$, $x_1 = bab^{-2}$, and $x_2 = b^2$. By a similar argument as when J is $P(-3,5,11)$, S does not satisfy the free factor property. $\square$

Theorem B. *Let J be $P(-3, 2q+1, 2r+1)$ pretzel knot with $1 \leqslant q \leqslant r$.*

(a) If J is $P(-3,3,2r+1)$, then $\pi(J)$ is bi-orderable.

(b) If J is $P(-3,5,2r+1)$ with $r > 3$ or $P(-3,2q+1,2r+1)$ with $q \geqslant 2$, then $\pi(J)$ is not bi-orderable.

Proof. When J is $P(-3,3,2r+1)$, J has Alexander polynomial $2t^2 - 5t + 2$ so J is rationally homologically fibered. Thus, by Lemma 6.11, the commutator subgroup of J is residually torsion-free nilpotent. Since Δ_J has two real positive roots, $\pi(J)$ is bi-orderable by Theorem 1.9.

When J is is $P(-3, 5, 2r+1)$ with $r > 3$ or $P(-3, 2q+1, 2r+1)$ with $q \geqslant 2$ then $N > 0$ so J is rationally homologically fibered and Δ_J has no real roots so $\pi(J)$ is not bi-orderable by Theorem 1.14 □

Table 6.1 summarizes the results we've found for the pretzel knots $P(-3, Q, R)$ where $Q = 2q+1$ and $R = 2r+1$. The shading of the cells, in each chart, indicate whether or not the knot's standard Seifert surface S satisfies the free factor property. Cells of knots with trivial Alexander polynomial are not shaded. The integers in each cell is the value of $N = \det(S_+) = \det(S_-)$, which is also the leading coefficient of the Alexander polynomial. If a pretzel knot's cell is shaded light blue and N is a prime power, then the knot's group has residually torsion-free nilpotent commutator subgroup. If in addition, $N < 0$, then the knot's group is bi-orderable.

6.5 A Family of Knots with Bi-Orderable Groups which Aren't Branched L-Space Knots

For each integer $q \geqslant 3$, let J_q be the pretzel knot $P(1-2q, 2q+1, 4q-3)$ so $p = -q$ and $r = 2q-2$

Lemma 6.16. *For all $q \geqslant 3$, the standard Seifert surface S of J_q satisfies the free factor property.*

		Values of R						
		3	5	7	9	11	13	15
Values of Q	3	-2	-2	-2	-2	-2	-2	-2
	5		-1	0	1	2	3	4
	7			2	4	6	8	10
	9				7	10	13	16
	11					14	18	22

Free Factor Property

No Free Factor Property

Table 6.1: Our results for some $P(-3, Q, R)$ where $Q = 2q + 1$ and $R = 2r + 1$. The integers in each cell are the values of N. Each cell is shaded light blue if the knot's standard Seifert surface satisfies the free factor property, and shaded dark red if the knot's standard Seifert surface does not satisfy the free factor property.

Proof. The knot J_3 is $P(-5,7,9)$. Thus, for J_3, S satisfies the free factor property by Lemma 6.10.

Assume $q \geqslant 4$. Define X, H, and K as above. After row reductions, $X/H[X,X]$ has presentation matrix

$$\begin{pmatrix} 1 & -(q+1) \\ 0 & -N \end{pmatrix}$$

where $N = -(q-1)^2$.

Let $C = -N = (q-1)^2$. Using Reidemeister-Schreier, we obtain the following basis.

$$\{ab^{-q-1}, bab^{-q-2}, \ldots, b^{C-q-2}ab^{1-C}, b^{C-q-1}a, b^{C-q}ab^{-1}, \ldots, b^{C-1}ab^{-q}, b^C\}$$

To simplify computations, we modify this basis by multiplying some of the elements by b^{-C} on the right, and obtain a free basis of $H[X,X]$, $\mathcal{B} = \{x_0, \ldots, x_C\}$ where $x_k = b^k ab^{-q-1-k}$ for $k = 0, \ldots, C-1$ and $x_C = b^C$.

We can rewrite α_H and β_H as

$$\begin{aligned}\alpha_H =&(b^{-1}a)^{q+1}a^{-q}\\ =&x_C^{-1}x_{C-1}x_C(x_{q-1}\cdots x_{i(q-2)-1}\cdots x_{q(q-2)-1})x_Cx_{q-2}x_{q-3}^{-1}x_C^{-1}\\ &(x_{(q-3)(q+1)}^{-1}x_{(q-4)(q+1)}^{-1}\cdots x_{(q-i)(q+1)}^{-1}\cdots x_0^{-1})\end{aligned}$$

and

$$\begin{aligned}\beta_H =&b^{2q-1}(a^{-1}b)^q\\ =&x_{q-2}^{-1}x_C^{-1}(x_{q(q-2)-1}^{-1}\cdots x_{q(q-i)-1}^{-1}\cdots x_{q-1}^{-1})x_C^{-1}x_{C-1}^{-1}x_C.\end{aligned}$$

Since $q \geqslant 4$, the generator x_0 appears once in the expression for α_H and does not appear in the expression for β_H. Also, since $q-2 < C-1$ and $qk-1 < C-1$ for all $k = 1, \ldots, q-2$, x_{C-1} only appears once in the expression for β_H.

Thus, x_{C-1} is a product of β_H, $x_1, \ldots, x_{C-2}, x_C$ and x_0 is a product of α_H, $x_1, \ldots, x_C$.

Therefore, the set $\{\alpha_H, \beta_H, x_1, \ldots, x_{C-2}, x_C\}$ is a free basis of $H[X,X]$ so H is a free factor of $H[X,X]$.

After row reductions, $X/K[X,X]$ has presentation matrix

$$\begin{pmatrix} 1 & -q \\ 0 & C \end{pmatrix}.$$

We obtain a free basis of $K[X,X]$, $\mathcal{B} = \{x_0, \ldots, x_C\}$ where $x_k = b^k a b^{-(q+k)}$ for $k = 0, \ldots, C-1$ and $x_C = b^C$.

We can rewrite α_K and β_K as

$$\begin{aligned} \alpha_K =& (ab^{-1})^q a^{-q+1} \\ =& (x_0 x_{q-1} x_{2(q-1)} \cdots x_{(q-2)(q-1)}) x_C x_0 x_C^{-1} \\ & (x_{q(q-2)}^{-1} x_{q(q-3)}^{-1} \cdots x_0^{-1}) \end{aligned}$$

and

$$\begin{aligned} \beta_K =& b^{2q-2}(ba^{-1})^{q+1} \\ =& x_{q-1}^{-1} x_0^{-1} x_C^{-1} (x_{(q-2)(q-1)}^{-1} x_{(q-3)(q-1)}^{-1} \cdots x_0^{-1}. \end{aligned}$$

The generator x_q appears once in the expression for α_K and does not appear in the expression for β_K. Also, x_C only appears once in the expression for β_K. Therefore, the set $\{\alpha_K, \beta_K, x_0, \ldots, x_{q-1}, x_{q+1}, \ldots, x_{C-1}\}$ is a free basis of $K[X, X]$ so K is a free factor of $K[X, X]$. Therefore, S satisfies the free factor property. $\square$

Theorem C. *For each integer $q \geqslant 3$ with $q-1$ is a prime power, $\pi(J_q)$ is bi-orderable, and $\Sigma_2(J_q)$ is not an L-space.*

Proof. By Lemma 6.16, J_q has a Seifert surface satisfying the free factor property. The Alexander polynomial of J_q is $Nt^2 + (1-2N)t + N$ where $N = -(q-1)^2$ so J_q is rationally homologically fibered and Δ_{J_q} has two positive real roots.

When $q-1$ is a prime power, $|\Delta_{J_q}(0)| = (q-1)^2$ is also a prime power. Therefore, when $q-1$ is a prime power, $\pi_1(M_{J_q})$ has residually torsion-free nilpotent commutator subgroup by Proposition 6.5, and $\pi(J_q)$ is bi-orderable by Proposition 6.7. Since $p = -q$, $\Sigma_2(J_q)$ is not an L-space by Corollary 1.22 for all $q \geqslant 3$. $\square$

Index

Bibliography

[1] P. C. Argyres and D. P. Kulkarni. *Fundamental groups for torus link complements* (2019). arXiv: `1904.10005 [math.GT]`.

[2] G. Baumslag. *Groups with the same lower Central Sequence as a Relatively Free Group. I The Groups*. Transactions of the American Mathematical Society 129, 2 (1967), 308–321.

[3] G. Baumslag. *Groups with the Same lower Central Sequence as a Relatively Free Group. II. Properties*. Transactions of the American Mathematical Society 142 (1969), 507–538.

[4] S. Boyer, C. M. Gordon, and L. Watson. *On L-spaces and left-orderable fundamental groups*. Mathematische Annalen 356, 4 (2013), 1213–1245.

[5] S. Boyer, D. Rolfsen, and B. Wiest. *Orderable 3-manifold groups*. en. Annales de l'Institut Fourier 55, 1 (2005), 243–288.

[6] E. M. Brown and R. H. Crowell. *The Augmentation Subgroup of a Link*. Journal of Mathematics and Mechanics 15, 6 (1966), 1065–1074.

[7] G. Burde and H. Zieschang. *Neuwirthsche Knoten und Flächenabbildungen*. Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg 31, 3 (1967), 239–246.

[8] G. Burde and H. Zieschang. *Knots*. Berlin, Germany: Walter de Gruyter, 1985.

[9] I. M. Chiswell, A. M. W. Glass, and J. S. Wilson. *Residual nilpotence and ordering in one-relator groups and knot groups*. Mathematical Proceedings of the Cambridge Philosophical Society 158, 2 (2015), 275–288.

[10] A. Clay, C. Desmarais, and P. Naylor. *Testing Bi-orderability of Knot Groups*. Canadian Mathematical Bulletin 59, 3 (2016), 472–482.

[11] A. Clay and D. Rolfsen. *Ordered groups, eigenvalues, knots, surgery and L-spaces*. Mathematical Proceedings of the Cambridge Philosophical Society 152, 1 (2012), 115–129.

[12] R. H. Crowell and H. F. Trotter. *A class of pretzel knots*. Duke Math. J. 30, 3 (1963), 373–377.

[13] D. Eisenbud, U. Hirsch, and W. Neumann. *Transverse foliations of Seifert bundles and self homeomorphism of the circle*. Commentarii Mathematici Helvetici 56, 1 (1981), 638–660.

[14] R. H. Fox. *Free Differential Calculus. I: Derivation in the Free Group Ring*. Annals of Mathematics 57, 3 (1953), 547–560.

[15] D. Gabai. *The Murasugi sum is a natural geometric operation. II*. In: *Combinatorial methods in topology and algebraic geometry (Rochester, N.Y., 1982)*. Vol. 44. Contemp. Math. Amer. Math. Soc., Providence, RI, 1985, 93–100.

[16] H. Goda and M. Ishiwata. *A classification of Seifert surfaces for some Pretzel links*. Kobe Journal of Mathematics 23 (2006).

[17] F. González-Acuña. *Dehn's construction on knots*. Bol. Soc. Mat. Mexicana 15, 2 (1970), 58–79.

[18] M. Hirasawa and K. Murasugi. *Fibred double torus knots which are band-sums of torus knots*. Osaka J. Math. 44, 1 (2007), 11–70.

[19] J. Howie and H. Short. *The Band-Sum Problem*. Journal of the London Mathematical Society s2-31, 3 (1985), 571–576. eprint: `https://londmathsoc.onlinelibrary.wiley.com/doi/pdf/10.1112/jlms/s2-31.3.571`.

[20] A. Issa and H. Turner. *Links all of whose cyclic branched covers are L-spaces*. 2020. arXiv: `2008.06127 [math.GT]`.

[21] T. Ito. *Alexander polynomial obstruction of bi-orderability for rationally homologically fibered knot groups*. arXiv: Geometric Topology (2016).

[22] M. Jankins and W. D. Neumann. *Rotation numbers of products of circle homeomorphisms*. Mathematische Annalen 271, 3 (1985), 381–400.

[23] J. Johnson. *Residual Torsion-Free Nilpotence, Bi-Orderability and Pretzel Knots*. 2020. arXiv: `2008.13353 [math.GT]`.

[24] J. Johnson. *Two-Bridge Knots and Residual Torsion-Free Nilpotence*. 2020. arXiv: `1912.08947 [math.GT]`.

[25] A. Karrass, W. Magnus, and D. Solitar. *Combinatorial group theory: presentations of groups in terms of generators and relations*. New York: Dover Publication, 1966.

[26] L. H. Kauffman. *State models and the jones polynomial*. Topology 26, 3 (1987), 395–407.

[27] L. H. Kauffman. *New invariants in the theory of knots*. en. In: *On the Geometry of Differentiable Manifolds Rome, 23-27 juin 1986*. Astérisque 163-164. Société mathématique de France, 1988.

[28] F. W. Levi. *Ordered groups.* Proceedings of the Indian Academy of Sciences - Section A 16, 4 (1942), 256.
[29] W. B. R. Lickorish and M. B. Thistlethwaite. *Some links with non-trivial polynomials and their crossing-numbers.* Commentarii Mathematici Helvetici 63, 1 (1988), 527–539.
[30] P. A. Linnell, A. H. Rhemtulla, and D. P. Rolfsen. *Invariant group orderings and Galois conjugates.* Journal of Algebra 319, 12 (2008), 4891–4898.
[31] P. Lisca and A. I. Stipsicz. *On the existence of tight contact structures on Seifert fibered 3-manifolds.* Duke Math. J. 148, 2 (2009), 175–209.
[32] L. Lyubich and K. Murasugi. *On zeros of the Alexander polynomial of an alternating knot.* Topology and its Applications 159, 1 (2012), 290–303.
[33] W. Magnus. *Beziehungen zwischen Gruppen und Idealen in einem speziellen Ring.* Mathematische Annalen 111 (1935), 259–280.
[34] E. J. Mayland. *Two-bridge knots have residually finite knot groups.* Proceedings of the Second International Conference on the Theory of Groups, Newman, M. F., edt. Lecture Notes in Mathematics 27, 5 (1974), 488–493.
[35] E. J. Mayland. *The Residual Finiteness of the Classical Knot Groups.* Canadian Journal of Mathematics 372, 5 (1975), 1092–1099.
[36] E. J. Mayland and K. Murasugi. *On a Structural Property of the Groups of Alternating Links.* Canadian Journal of Mathematics 28, 3 (1976), 568–588.
[37] J. M. Montesinos. *Surgery on Links and Double Branched Covers of S^3.* In: *Knots, Groups and 3-Manifolds (AM-84): Papers Dedicated to the Memory of R.H. Fox. (AM-84).* Princeton University Press, 1975, 227–260.
[38] K. Murasugi. *Jones polynomials and classical conjectures in knot theory.* Topology 26, 2 (1987), 187–194.
[39] K. Murasugi. *Knot Theory and Its Applications.* Modern Birkhäuser Classics. Birkhäuser Basel, 1996.
[40] R. Naimi. *Foliations transverse to fibers of Seifert manifolds.* Commentarii Mathematici Helvetici 69, 1 (1994), 155–162.
[41] B. H. Neumann. *On Ordered Groups.* American Journal of Mathematics 71, 1 (1949), 1–18.
[42] P. Ozsváth and Z. Szabó. *Holomorphic disks and genus bounds.* Geom. Topol. 8, 1 (2004), 311–334.

[43] B. Perron and D. Rolfsen. *On orderability of fibred knot groups*. Mathematical Proceedings of the Cambridge Philosophical Society 135, 1 (2003), 147–153.

[44] K. Reidemeister. *Knotentheorie*. Vol. 1. Ergebnisse der Mathematik und ihrer Grenzgebiete. 2. Folge. Springer-Verlag Berlin Heidelberg, 1932.

[45] D. Rolfsen. *Knots and links*. Berkeley, CA, 1976.

[46] D. Rolfsen. *Ordered groups as a tensor category*. Pacific Journal of Mathematics 294, 1 (2018), 181–194.

[47] O. Schreier. *Die Untergruppen der freien Gruppen*. Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg 5, 1 (1927), 161–183.

[48] H. Schubert. *Knoten mit zwei Brücken*. Mathematische Zeitschrift 65, 1 (1956), 133–170.

[49] J. Stallings. *On fibering certain 3-manifolds*. In: 1961.

[50] M. Teragaito. *Fourfold cyclic branched covers of genus one two-bridge knots are L-spaces*. Boletín de la Sociedad Matemática Mexicana 20, 2 (2014), 391–403.

[51] M. B. Thistlethwaite. *A spanning tree expansion of the jones polynomial*. Topology 26, 3 (1987), 297–309.

[52] A. A. Vinogradov. *On the free product of ordered groups*. Matematicheskii Sbornik. Novaya Seriya 25, 67 (1 1949), 163–168.

[53] T. Yamada. *A family of bi-orderable non-fibered 2-bridge knot groups*. Journal of Knot Theory and Its Ramifications 26, 04 (2017), 1750011.